On the Convergence of the Integral Representation of $\zeta(2)$

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Abstract

An irrational number is a number that cannot be expressed as a fraction $\frac{p}{q}$ for any integers p and q, $q \neq 0$. It follows that irrational numbers have decimal expansions that neither terminate nor become periodic. This makes proofs of irrationality very difficult. Examples of proven irrational numbers include $\sqrt{2}, \pi, e$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$, but there remains many numbers for which it is not known whether or not they are irrational, such as Euler's constant. One can find many methods used to evaluate $\zeta(2)$ at least some of which depend on the convergence of $\int_0^1\int_0^1 \frac{1}{1-xy} dydx$. A proof of the convergence of the integral representation of $\zeta(2)$ is presented.

Background

Consider the infinite sum $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \ldots$. Euler himself worked on this problem and obtained approximation formulas that allowed him to determine the sum to the first several decimal places. In 1734, at the age of 28 Euler proved $\zeta(2) = \frac{\pi^2}{6}$. Here is a modification of one of Euler's proofs [1]. The sine function can be represented as the power series,

$$\sin x = x - \frac{x^3}{3\cdot 2} + \frac{x^5}{5\cdot 4\cdot 3\cdot 2} - \frac{x^7}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \ldots .$$

Divide by x and replace x with $\sqrt{x}$ we have,

$$\frac{\sin\sqrt{x}}{\sqrt{x}} = 1 - \frac{x}{3\cdot 2} + \frac{x^2}{5\cdot 4\cdot 3\cdot 2} - \frac{x^3}{7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2} + \ldots .$$

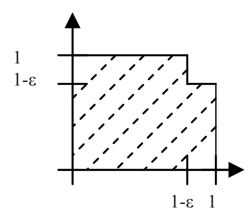
We call this function f and note that the roots of f are $\pi^2, 4\pi^2, 9\pi^2, \ldots$. Adding the reciprocals of the roots of a polynomial results in the negative of the ratio of the linear coefficient to the constant coefficient, or if $(x-r_1)(x-r_2)\cdots(x-r_n) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ then $\frac{1}{r_1} + \frac{1}{r_2} + \cdots + \frac{1}{r_n} = -\frac{a_1}{a_0}$.

Applying this to f produces, $\frac{1}{6} = \frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{16\pi^2} + \cdots$. Hence, Euler concluded that the sum is $\frac{\pi^2}{6}$. It is important to note that not all power series share all of the properties of polynomials.

The convergence of $\int_0^1\int_0^1 \frac{1}{1-xy}\,dydx$

One can find many methods used to evaluate $\zeta(2)$, at least some of which depend on the convergence of $\int_0^1\int_0^1 \frac{1}{1-xy}\,dydx$ [2]. A proof is presented here.

First, define $I_\varepsilon = \iint_{R_\varepsilon} \frac{1}{1-xy}\,dA$, where $0 < \varepsilon < 1$, R_ε shown.



Then, if $\lim_{\varepsilon \to 0^+} I_\varepsilon = I,\ I \in R$ then $\int_0^1\int_0^1 \frac{1}{1-xy}\,dxdy$ converges to *I*.

Write, $I_\varepsilon = \int_0^{1-\varepsilon}\int_0^{1-\varepsilon} \frac{1}{1-xy}\,dxdy + \int_0^{1-\varepsilon}\int_{1-\varepsilon}^{1} \frac{1}{1-xy}\,dxdy + \int_{1-\varepsilon}^{1}\int_0^{1-\varepsilon} \frac{1}{1-xy}\,dxdy$, by symmetry define $S_\varepsilon = \int_0^{1-\varepsilon}\int_{1-\varepsilon}^{1} \frac{1}{1-xy}dxdy = \int_{1-\varepsilon}^{1}\int_0^{1-\varepsilon} \frac{1}{1-xy}dxdy$

and define $J_\varepsilon = \int_0^{1-\varepsilon}\int_0^{1-\varepsilon} \frac{1}{1-xy}dxdy$. Thus, $I_\varepsilon = J_\varepsilon + 2S_\varepsilon$.

Begin with S_ε.

$$0 < S_\varepsilon = \int_0^{1-\varepsilon}\int_{1-\varepsilon}^{1} \frac{1}{1-xy}dxdy \le \int_0^{1-\varepsilon}\left(\int_{1-\varepsilon}^{1} \frac{1}{1-y}dx\right)dy = \int_0^{1-\varepsilon} \frac{\varepsilon}{1-y}dy = -\varepsilon \ln \varepsilon$$

and, $\lim_{\varepsilon \to 0^+}(-\varepsilon \ln \varepsilon) = 0$. Thus $\lim_{\varepsilon \to 0+} S_\varepsilon = 0$ Now, $S_\varepsilon{}'$ takes on a maximum value of $\frac{1}{e}$ and $S_\varepsilon\left(\frac{1}{e}\right) = \frac{1}{e}$, hence, $0 < S_\varepsilon \le \frac{1}{e}$ and $\lim_{\varepsilon \to 0+} S_\varepsilon = 0$.

Continuing with J_ε. We note that

i) R_ε is increasing as $\varepsilon \to 0^+$ so I_ε is increasing as $\varepsilon \to 0^+$,

ii) $[0,\ 1-\varepsilon]$x$[0,\ 1-\varepsilon]$ is increasing as $\varepsilon \to 0^+$ so J_ε is increasing as $\varepsilon \to 0^+$ and

iii) for all $(x, y) \in R_\varepsilon$, $|xy| \le (1-\varepsilon)$, so $\frac{1}{1-xy} = \sum_{n=0}^{\infty}(xy)^n$.

Then, $J_\varepsilon = \int_0^{1-\varepsilon}\int_0^{1-\varepsilon} \frac{1}{1-xy}dxdy = \int_0^{1-\varepsilon}\int_0^{1-\varepsilon}\left(\sum_{n=0}^{\infty} x^n y^n\right)dxdy$. **(1)**

Since $\sum_{n=0}^{\infty} x^n y^n$ is absolutely convergent on $[0,\ 1-\varepsilon]$ x $[0,\ 1-\varepsilon]$, **(1)** becomes,

$$\int_0^{1-\varepsilon}\left(\sum_{n=0}^{\infty}\int_0^{1-\varepsilon}(x^n y^n)dx\right)dy = \int_0^{1-\varepsilon}\left(\sum_{n=0}^{\infty} y^n \frac{(1-\varepsilon)^{n+1}}{n+1}\right)dy$$

$$= \sum_{n=0}^{\infty}\left(\int_0^{1-\varepsilon} y^n dy\right)\frac{(1-\varepsilon)^{n+1}}{n+1} = \sum_{n=0}^{\infty}\frac{(1-\varepsilon)^{2(n+1)}}{(n+1)^2} = \sum_{n=1}^{\infty}\frac{(1-\varepsilon)^{2n}}{n^2} .$$

Thus, $J_\varepsilon = \sum_{n=1}^{\infty} \frac{(1-\varepsilon)^{2n}}{n^2} < \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$.

Since, J_ε is increasing as $\varepsilon \to 0^+$ and J_ε is bounded by $\frac{\pi^2}{6}$ we have

$J_\varepsilon \to L \le \frac{\pi^2}{6}$, and $0 \le S_\varepsilon \le -\varepsilon \ln \varepsilon \to 0$ as $\varepsilon \to 0^+$ then,

$$I_\varepsilon \to L + 0 = L \le \frac{\pi^2}{6}.$$

Now show $L = \frac{\pi^2}{6}$.

Since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, then given $\alpha > 0$ there exists $N \in \boldsymbol{N}$ such that

$$\sum_{n=1}^{N} \frac{1}{n^2} > \frac{\pi^2}{6} - \frac{\alpha}{2}.$$

Given N, there exists $\delta > 0$ such that

if $0 < \varepsilon < \delta$, then $(1-\varepsilon)^{2N} > 1 - \frac{\alpha}{2\left(\frac{\pi^2}{6}\right)}$.

So if $\varepsilon < \delta$ then,

$$L > J_\varepsilon = \sum_{n=1}^{\infty} \frac{(1-\varepsilon)^{2n}}{n^2} > \sum_{n=1}^{N} \frac{(1-\varepsilon)^{2n}}{n^2} > (1-\varepsilon)^{2N} \sum_{n=1}^{N} \frac{1}{n^2} >$$

$$\left[1 - \frac{\alpha}{2\left(\frac{\pi^2}{6}\right)}\right]\left(\frac{\pi^2}{6} - \frac{\alpha}{2}\right) = \frac{\pi^2}{6} - \frac{\alpha}{2} - \frac{\alpha}{2} + \frac{3\alpha^2}{2\pi^2} > \frac{\pi^2}{6} - \alpha$$

.

Thus, for all $\alpha > 0$, $L > \frac{\pi^2}{6} - \alpha$, so $L \ge \frac{\pi^2}{6}$.

Hence $L = \frac{\pi^2}{6}$ and $\int_0^1\int_0^1 \frac{1}{1-xy} dxdy = \frac{\pi^2}{6}$.

Conclusion

We now see that [2]

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dydx =$$

$$\int_0^1\int_0^1 \sum_{i\geq 0} x^i y^i\,dydx = \sum_{i\geq 0}\int_0^1 x^i\,dx \int_0^1 y^i\,dy = \sum_{i\geq 0}\left[\frac{x^{i+1}}{i+1}\right]_0^1 \left[\frac{y^{i+1}}{i+1}\right]_0^1$$

$$= \sum_{i\geq 0}\frac{1}{(i+1)^2} = \zeta(2).$$

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References

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2. D. Huylebrouck, "Similarities in Irrationality Proofs for π, $\ln 2$, $\zeta(2)$, and $\zeta(3)$." Amer. Math. Monthly 108 (2001), 222-231.