Calculation of Potential Flow Around An Elliptic Cylinder Using Boundary Element Method

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Abstract

In this paper, a direct boundary element method is applied for calculating the incompressible potential flow field (i.e. velocity distribution) around an elliptic cylinder. To check the accuracy of the method, the computed flow velocity is compared with the analytical solution for the flow over the boundary of an elliptic cylinder. It can be seen that the output given by the boundary element method is approximately the same as the analytical result obtained for an elliptic cylinder.

Elliptic Coordinates

Let $z = c \cosh \zeta$ (1)

where $z = x + i y$ and $\zeta = \xi + i \eta$

be a transformation from the ζ–plane to the z–plane.

Then (1) implies the transformation equations

$x = c \cosh \xi \cos \eta$ (2)

$y = c \sinh \xi \sin \eta$ (3)

Squaring and adding (2) and (3) , we get

$$\frac{x^2}{c^2 \cosh^2 \xi} + \frac{y^2}{c^2 \sinh^2 \xi} = 1 \quad (4)$$

From (4) , it is clear that if ξ has the constant value ξ_0, the point (x, y) lies on an ellipse whose semi – axes a and b are given by

$a = c \cosh \xi_0$ and $b = c \sinh \xi_0$ (5)

and therefore

$$a^2 - b^2 = c^2 (\cosh^2 \xi_0 - \sinh^2 \xi_0) = c^2 \quad (6)$$

The ellipses corresponding to constant values of ξ are therefore confocal , the distance between the foci being $2c$.

Now for the ellipse $\xi = \xi_0$, (5) gives

$$\xi_0 = \tanh^{-1}\frac{b}{a} = \frac{1}{2}\ln\left(\frac{a+b}{a-b}\right) \qquad (7)$$

Equation (7) determines the parameter ξ_0 in terms of semi – axes a and b .

Flow Past An Elliptic Cylinder

Let there be a circular cylinder of radius $|\zeta| = \frac{a+b}{2}$ with centre at the origin of the ζ – plane. Let a uniform stream of velocity U in the positive direction of ξ – axis be flowing past the cylinder (Figure I) .

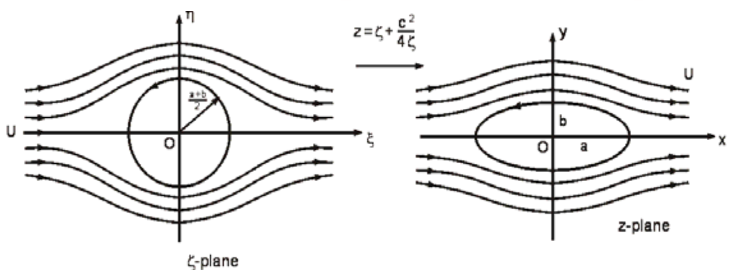


Figure I

We know that the complex velocity potential for the flow past a circular cylinder of radius $|\zeta| = \frac{a+b}{2}$ in the ζ – plane is given by circle theorem

$$w(\zeta) = -U\left[\zeta + \frac{(a+b)^2}{4\zeta}\right] \qquad (8)$$

The Joukowski transformation

$$z = \zeta + \frac{c^2}{4\zeta}, \quad c^2 = a^2 - b^2 \qquad (9)$$

transforms the circle $|\zeta| = \frac{a+b}{2}$ in the ζ – plane onto an ellipse of semi – axes a and b , with centre at the origin in the z – plane .

Since $z = x + i y$ and $\zeta = \xi + i \eta$

Then $$x = \xi\left(1 + \frac{c^2}{4(\xi^2+\eta^2)}\right), \quad y = \eta\left(1 - \frac{c^2}{4(\xi^2+\eta^2)}\right)$$

are the transformation equations from the ζ – plane to z – plane .

The complex velocity potential $w(z)$ for flow past an elliptic cylinder having cross – section $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in the z – plane is found by eliminating ζ from (8) and (9) . Solving (9) , we get

$$4\zeta^2 - 4z\zeta + c^2 = 0$$

Then $$\zeta = \frac{z \pm \sqrt{z^2 - c^2}}{2}$$

We take the positive sign with the radical so as to obtain the transformation which maps the region outside the circle in the ζ – plane onto the region outside the ellipse in the z – plane .

Thus $$\zeta = \frac{1}{2}\left(z + \sqrt{z^2 - c^2}\right) \qquad (10)$$

From (8) and (10) , we get

$$w(z) = -U\left(\frac{a+b}{2}\right)\left[\frac{z+\sqrt{z^2-c^2}}{a+b} + \frac{z-\sqrt{z^2-c^2}}{a-b}\right] \qquad (11)$$

is the required complex velocity potential .

The form of complex velocity potential can , however , be simplified by means of elliptic coordinates , which takes the form

$$w(\zeta) = -U c e^{\xi_0} \cosh(\zeta - \xi_0) \qquad (12)$$

To find the speed V , the complex velocity is given by

$$\frac{dw}{dz} = \frac{dw}{d\zeta}\frac{d\zeta}{dz}$$

which in elliptic coordinates becomes

$$\frac{dw}{dz} = -U e^{\xi_0} \frac{\sinh(\xi-\xi_0)\cos\eta + i\cosh(\xi-\xi_0)\sin\eta}{\sinh\xi\cos\eta + i\cosh\xi\sin\eta}$$

Thus

$$V = \left| \frac{dw}{dz} \right|$$

$$= U e^{\xi_0} \frac{\sqrt{\sinh^2(\xi - \xi_0)\cos^2\eta + \cosh^2(\xi - \xi_0)\sin^2\eta}}{\sqrt{\sinh^2\xi\cos^2\eta + \cosh^2\xi\sin^2\eta}} \qquad (13)$$

Squaring and after simplification, (13) takes the form

$$V^2 = U^2 e^{2\xi_0} \left[\frac{\sinh^2(\xi - \xi_0) + \sin^2\eta}{\sinh^2\xi + \sin^2\eta} \right]$$

Since $e^{2\xi_0} = \frac{a+b}{a-b}$, therefore

$$V^2 = U^2 (a+b)^2 \left[\frac{\sin^2\eta}{b^2 + (a^2 - b^2)\sin^2\eta} \right] \qquad (14)$$

Dividing (3) by (2)

$$\frac{y}{x} = \tanh\xi \tan\eta \qquad (15)$$

On the surface of the cylinder, $\xi = \xi_0$, (7) gives $\tanh\xi_0 = \frac{b}{a}$

From (15), it follows that

$$\sin\eta = \frac{a\,y}{\sqrt{b^2 x^2 + a^2 y^2}} \qquad (16)$$

and $$\cos\eta = \frac{b\,x}{\sqrt{b^2 x^2 + a^2 y^2}} \qquad (17)$$

Using (16), after simplification (14) takes the form

$$V = U(a+b)\frac{a\,y}{\sqrt{b^4 x^2 + a^4 y^2}} \qquad (18)$$

which is the analytical formula for calculating the velocity distribution past an elliptic cylinder in terms of a and b.

Velocity Distribution

Boundary element methods can be used to solve two-dimensional interior or exterior flow problems. Consider the flow past an elliptic cylinder of semi axes a and b with center at the origin and let the onset flow be the uniform stream with velocity U in the positive direction of the x-axis as shown in Figure II.

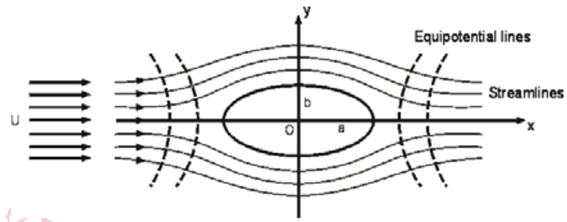


Figure II

The condition to be satisfied on the boundary of an elliptic cylinder is

$$\frac{\partial \phi_{e.c}}{\partial n} = -\frac{\partial \phi_{u.s}}{\partial n} \qquad (19)$$

where $\phi_{u.s}$ is the velocity potential of uniform stream and $\phi_{e.c}$ is the velocity potential at the surface of the elliptic cylinder.

But the velocity potential of the uniform stream is given as

$$\phi_{u.s} = -U x$$

Then $$\frac{\partial \phi_{u.s}}{\partial n} = -U\frac{\partial x}{\partial n} = -U\left(\hat{n}\,.\,\hat{i}\right) \qquad (20)$$

Thus from (19) and (20), we get

$$\frac{\partial \phi_{e.c}}{\partial n} = U\left(\hat{n}\,.\,\hat{i}\right) \qquad (21)$$

The equation of the boundary of the elliptic cylinder is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad (22)$$

Thus from (21), we get

$$\frac{\partial \phi_{e.c}}{\partial n} = U\frac{x\,b^2}{\sqrt{b^4 x^2 + a^4 y^2}} \qquad (23)$$

Equation (23) is the boundary condition which must be satisfied over the boundary of an elliptic cylinder.

Now for the approximation of the boundary of the elliptic cylinder, the coordinates of the extreme points of the boundary elements can be generated within the computer program as follows:

Divide the boundary of the elliptic cylinder into m elements in the clockwise direction by using the formula

$$\theta_k = [(m + 3) - 2k]\,\pi / m, \quad k = 1, 2, \ldots, m \qquad (24)$$

Then the coordinates of the extreme points of these m elements are calculated from

$$\left.\begin{array}{l} x_k = a \cos \theta_k \\ y_k = b \sin \theta_k \end{array}\right\}, \quad k = 1, 2, \ldots, m \qquad (25)$$

Take $m = 8$, $a = 2$ and $b = 1$.

First consider the case of constant boundary elements where there is only one node at the middle of the element and the potential ϕ and the potential derivative $\frac{\partial \phi}{\partial n}$ are constant over each element and equal to the value at the middle node of the element.

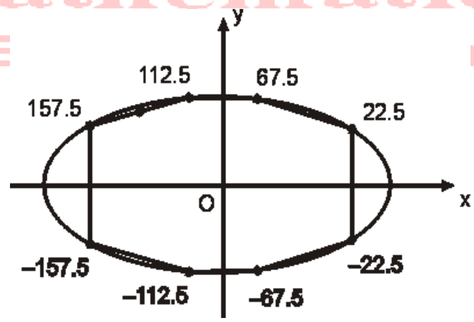


Figure III

Figure III shows the discretization of an elliptic cylinder into 8 constant boundary element. The coordinates of the middle node of each boundary element are given by

$$\left.\begin{array}{l} x_m = (x_k + x_{k+1}) / 2 \\ y_m = (y_k + y_{k+1}) / 2 \end{array}\right\} \quad k, \; m = 1, 2, \ldots, 8 \qquad (26)$$

and therefore the boundary condition (23) in this case takes the form

$$\frac{\partial \phi_{e.c}}{\partial n} = U \frac{x_m b^2}{\sqrt{b^4 x_m^2 + a^4 y_m^2}} .$$

The velocity U of the uniform stream is also taken as unity.

Table I
The comparison of the analytical and computed velocity distributions over the boundary of the elliptic cylinder for 8 constant boundary elements.

Element	x – coordinate	y – coordinate	$R=\sqrt{x^2+y^2}$	Computed Velocity	Analytical Velocity
1	-1.58	.33	.16106E+01	.10777E+01	.95693E+00
2	-.65	.79	.10240E+01	.14989E+01	.14688E+01
3	.65	.79	.10240E+01	.14989E+01	.14688E+01
4	1.58	.33	.16106E+01	.10777E+01	.95693E+00
5	1.58	-.33	.16106E+01	.10777E+01	.95693E+00
6	.65	-.79	.10240E+01	.14989E+01	.14688E+01
7	-.65	-.79	.10240E+01	.14989E+01	.14688E+01
8	-1.58	-.33	.16106E+01	.10777E+01	.95693E+00

Table II
The improvement gained by using 16 constant boundary elements

Element	x – coordinate	y – coordinate	$R=\sqrt{x^2+y^2}$	Computed Velocity	Analytical Velocity
1	-1.89	.19	.18962E+01	.58244E+00	.55447E+00
2	-1.60	.53	.16866E+01	.12226E+01	.12010E+01
3	-1.07	.80	.13350E+01	.14319E+01	.14227E+01
4	-.38	.94	.10154E+01	.14984E+01	.14926E+01
5	.38	.94	.10154E+01	.14984E+01	.14926E+01
6	1.07	.80	.13350E+01	.14319E+01	.14227E+01
7	1.60	.53	.16866E+01	.12226E+01	.12010E+01
8	1.89	.19	.18962E+01	.58244E+00	.55447E+00
9	1.89	-.19	.18962E+01	.58244E+00	.55447E+00
10	1.60	-.53	.16866E+01	.12226E+01	.12010E+01
11	1.07	-.80	.13350E+01	.14319E+01	.14227E+01
12	.38	-.94	.10154E+01	.14984E+01	.14926E+01
13	-.38	-.94	.10154E+01	.14984E+01	.14926E+01
14	-1.07	-.80	.13350E+01	.14319E+01	.14227E+01
15	-1.60	-.53	.16866E+01	.12226E+01	.12010E+01
16	-1.89	-.19	.18962E+01	.58244E+00	.55447E+00

Table III
The improvement gained by using 32 constant boundary elements

Element	x – coordinate	y – coordinate	$R = \sqrt{x^2 + y^2}$	Comp. Velocity	Ana. Velocity
1	-1.97	.10	.19736E+01	.29449E+00	.28991E+00
2	-1.90	.29	.19172E+01	.78642E+00	.77805E+00
3	-1.75	.47	.18082E+01	.11020E+01	.10954E+01
4	-1.53	.63	.16551E+01	.12852E+01	.12810E+01
5	-1.26	.77	.14714E+01	.13904E+01	.13877E+01
6	-.93	.87	.12786E+01	.14510E+01	.14491E+01
7	-.57	.95	.11085E+01	.14845E+01	.14830E+01
8	-.19	.99	.10046E+01	.14995E+01	.14982E+01
9	.19	.99	.10046E+01	.14995E+01	.14982E+01
10	.57	.95	.11085E+01	.14845E+01	.14830E+01
11	.93	.87	.12786E+01	.14510E+01	.14491E+01
12	1.26	.77	.14714E+01	.13904E+01	.13877E+01
13	1.53	.63	.16551E+01	.12852E+01	.12810E+01
14	1.75	.47	.18082E+01	.11020E+01	.10954E+01
15	1.90	.29	.19172E+01	.78641E+00	.77805E+00
16	1.97	.10	.19736E+01	.29449E+00	.28990E+00
17	1.97	-.10	.19736E+01	.29449E+00	.28990E+00
18	1.90	-.29	.19172E+01	.78641E+00	.77805E+00
19	1.75	-.47	.18082E+01	.11020E+01	.10954E+01
20	1.53	-.63	.16551E+01	.12852E+01	.12810E+01
21	1.26	-.77	.14714E+01	.13904E+01	.13877E+01
22	.93	-.87	.12786E+01	.14510E+01	.14491E+01
23	.57	-.95	.11085E+01	.14845E+01	.14830E+01
24	.19	-.99	.10046E+01	.14995E+01	.14982E+01
25	-.19	-.99	.10046E+01	.14995E+01	.14982E+01
26	-.57	-.95	.11085E+01	.14845E+01	.14830E+01
27	-.93	-.87	.12786E+01	.14510E+01	.14491E+01
28	-1.26	-.77	.14714E+01	.13904E+01	.13877E+01
29	-1.53	-.63	.16551E+01	.12852E+01	.12810E+01
30	-1.75	-.47	.18082E+01	.11020E+01	.10954E+01
31	-1.90	-.29	.19172E+01	.78642E+00	.77805E+00
32	-1.97	-.10	.19736E+01	.29448E+00	.28990E+00

Table IV
The comparison of the analytical and computed velocities at the mid points of 8 linear boundary elements over an elliptic cylinder.

Element	x – coordinate	y – coordinate	R = $\sqrt{x^2 + y^2}$	Computed Velocity	Analytical Velocity
1	-1.71	.35	.17433E+01	.87357E+00	.95693E+00
2	-.71	.85	.11084E+01	.14583E+01	.14688E+01
3	.71	.85	.11084E+01	.14583E+01	.14688E+01
4	1.71	.35	.17433E+01	.87357E+00	.95693E+00
5	1.71	-.35	.17433E+01	.87357E+00	.95693E+00
6	.71	-.85	.11084E+01	.14583E+01	.14688E+01
7	-.71	-.85	.11084E+01	.14583E+01	.14688E+01
8	-1.71	-.35	.17433E+01	.87357E+00	.95693E+00

Table V
The improvement gained by using 16 linear boundary elements

Element	x – coordinate	y – coordinate	R = $\sqrt{x^2 + y^2}$	Computed Velocity	Analytical Velocity
1	-1.92	.19	.19334E+01	.51847E+00	.55447E+00
2	-1.63	.54	.17196E+01	.11710E+01	.12010E+01
3	-1.09	.82	.13611E+01	.14180E+01	.14227E+01
4	-.38	.96	.10353E+01	.14950E+01	.14926E+01
5	.38	.96	.10353E+01	.14950E+01	.14926E+01
6	1.09	.82	.13611E+01	.14180E+01	.14227E+01
7	1.63	.54	.17196E+01	.11710E+01	.12010E+01
8	1.92	.19	.19334E+01	.51848E+00	.55447E+00
9	1.92	-.19	.19334E+01	.51847E+00	.55447E+00
10	1.63	-.54	.17196E+01	.11710E+01	.12010E+01
11	1.09	-.82	.13611E+01	.14180E+01	.14227E+01
12	.38	-.96	.10353E+01	.14950E+01	.14926E+01
13	-.38	-.96	.10353E+01	.14950E+01	.14926E+01
14	-1.09	-.82	.13611E+01	.14180E+01	.14227E+01
15	-1.63	-.54	.17196E+01	.11710E+01	.12010E+01
16	-1.92	-.19	.19334E+01	.51847E+00	.55447E+00

Table VI
The improvement gained by using 32 linear boundary elements

Element	x – coordinate	y – coordinate	$R = \sqrt{x^2 + y^2}$	Computed Velocity	Analytical Velocity
1	-1.98	.10	.19832E+01	.28201E+00	.28991E+00
2	-1.90	.29	.19264E+01	.76380E+00	.77805E+00
3	-1.76	.47	.18170E+01	.10855E+01	.10954E+01
4	-1.54	.63	.16631E+01	.12762E+01	.12810E+01
5	-1.26	.77	.14786E+01	.13860E+01	.13877E+01
6	-.94	.88	.12848E+01	.14490E+01	.14491E+01
7	-.58	.95	.11139E+01	.14836E+01	.14830E+01
8	-.20	.99	.10094E+01	.14990E+01	.14982E+01
9	.20	.99	.10094E+01	.14990E+01	.14982E+01
10	.58	.95	.11139E+01	.14836E+01	.14830E+01
11	.94	.88	.12848E+01	.14490E+01	.14491E+01
12	1.26	.77	.14786E+01	.13860E+01	.13877E+01
13	1.54	.63	.16631E+01	.12762E+01	.12810E+01
14	1.76	.47	.18170E+01	.10855E+01	.10954E+01
15	1.90	.29	.19264E+01	.76380E+00	.77805E+00
16	1.98	.10	.19832E+01	.28201E+00	.28990E+00
17	1.98	-.10	.19832E+01	.28201E+00	.28990E+00
18	1.90	-.29	.19264E+01	.76380E+00	.77805E+00
19	1.76	-.47	.18170E+01	.10855E+01	.10954E+01
20	1.54	-.63	.16631E+01	.12762E+01	.12810E+01
21	1.26	-.77	.14786E+01	.13860E+01	.13877E+01
22	.94	-.88	.12848E+01	.14490E+01	.14491E+01
23	.58	-.95	.11139E+01	.14836E+01	.14830E+01
24	.20	-.99	.10094E+01	.14990E+01	.14982E+01
25	-.20	-.99	.10094E+01	.14990E+01	.14982E+01
26	-.58	-.95	.11139E+01	.14836E+01	.14830E+01
27	-.94	-.88	.12848E+01	.14490E+01	.14491E+01
28	-1.26	-.77	.14786E+01	.13860E+01	.13877E+01
29	-1.54	-.63	.16631E+01	.12762E+01	.12810E+01
30	-1.76	-.47	.18170E+01	.10855E+01	.10954E+01
31	-1.90	-.29	.19264E+01	.76381E+00	.77805E+00
32	-1.98	-.10	.19832E+01	.28201E+00	.28990E+00

Figure IV: Comparison of computed and analytical velocity distributions over the surface of an elliptic cylinder using 8 boundary elements with constant variation.

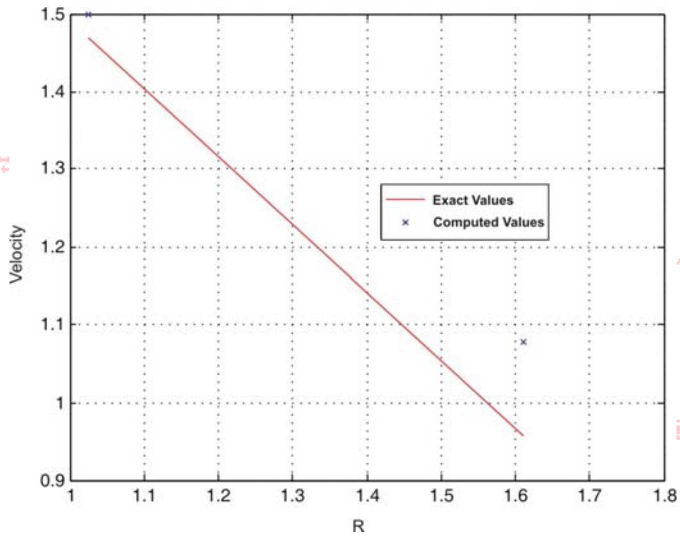


Figure V: Comparison of computed and analytical velocity distributions over the surface of an elliptic cylinder using 16 boundary elements with constant variation.

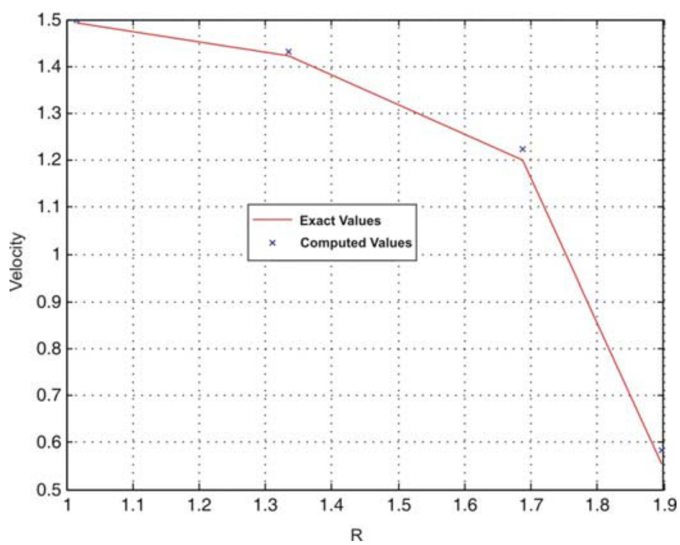
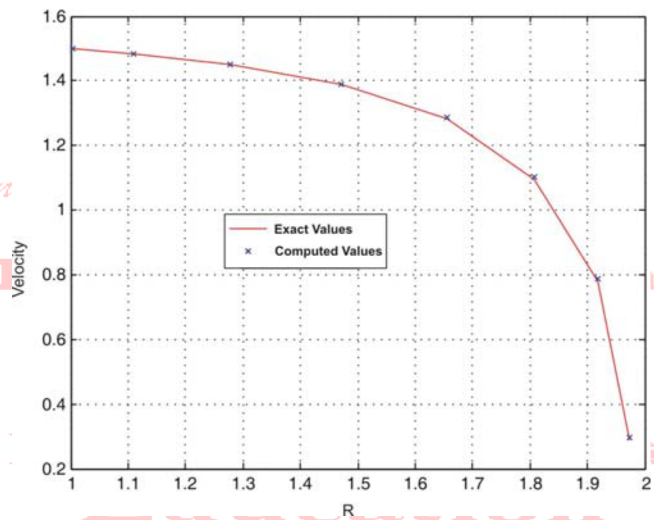


Figure VI: Comparison of computed and analytical velocity distributions over the surface of an elliptic cylinder using 32 boundary elements with constant variation.



Consider next the case where the boundary of an elliptic cylinder is divided into linear elements. In this case the nodes where the boundary conditions are specified are at the intersection of the elements. The boundary of the cylinder can be divided into elements by the formula

$$\theta_k = [(m + 2) - 2k]\, \pi / m, \quad k = 1, 2, \ldots, m \tag{27}$$

Figure VII shows the discretization of an elliptic cylinder into 8 linear boundary elements.

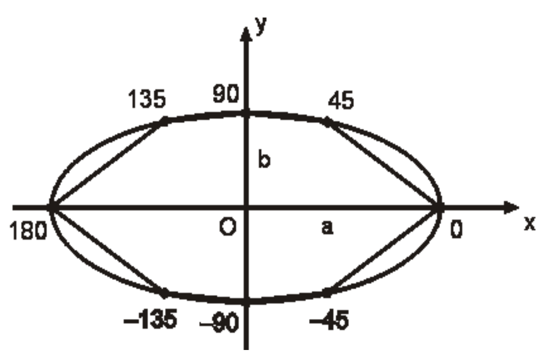


Figure VII

The coordinates of the extreme points of the elements and the coordinates of the mid-point of each element where the velocity will be calculated can be found from (23) and (24) respectively. The reason for distributing the elements in this way is such that the velocities are calculated at the same values of θ in both figures III and IV, so that the calculated results could be easily compared.

Figure VIII: Comparison of computed and analytical velocity distributions over the surface of an elliptic cylinder using 8 boundary elements with linear variation.

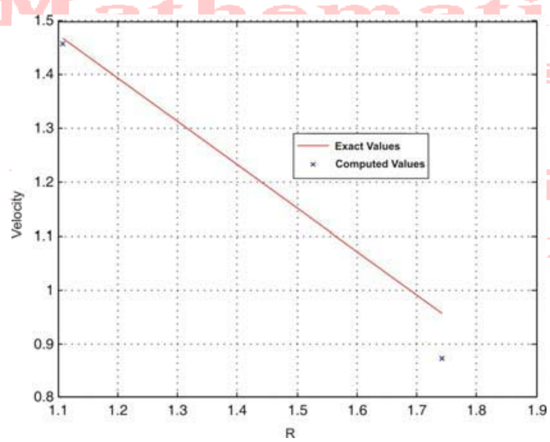


Figure IX: Comparison of computed and analytical velocity distributions over the surface of an elliptic cylinder using 16 boundary elements with linear variation.

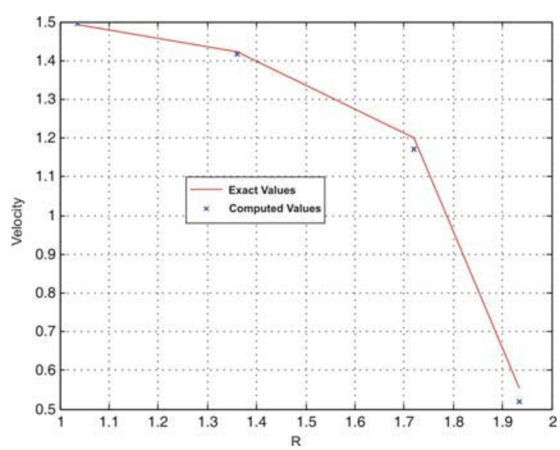
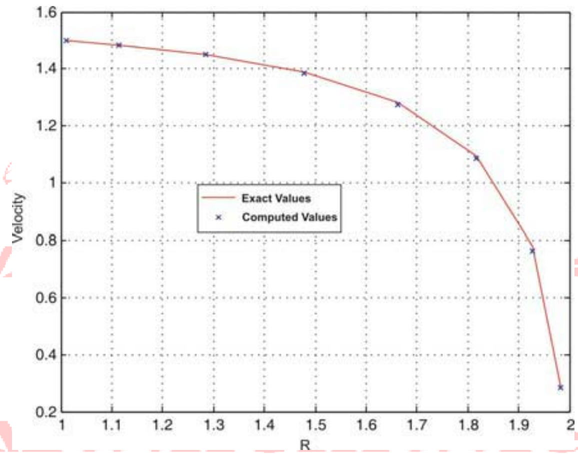


Figure X: Comparison of computed and analytical velocity distributions over the surface of an elliptic cylinder using 32 boundary elements with linear variation.



Conclusion

A direct boundary element method has been used for the calculation of incompressible potential flow around two–dimensional body i.e. an elliptic cylinder. The computed velocity distribution obtained using this method is compared with the analytical solutions for flows over the boundary of an elliptic cylinder using 8,16 and 32 constant and linear boundary elements. It is found that the results obtained with the direct boundary element method for the flow field calculations are in excellent agreement with the analytical results for the bodies under consideration.

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