Using Possibilities to Compare the Intelligence of Student Groups in the van Hiele Level Theory

Steve C. Perdikaris, Ph.D. †

Abstract

The intelligence of student groups is a topic of importance in mathematics education. The purpose of this work is to compare the intelligence of student groups in the van Hiele level theory of geometric thinking. The sample consists of 50 students in groups A, B and C with 20, 21 and 9 students respectively. All students have the same mathematical background. Since the ambiguity (aggregate possibilistic uncertainty) of student groups can be seen as the student group intelligence a measure of ambiguity is used to compare the student group intelligence in a geometric task. This work provides support for the strong effect of student group intelligence on student group performance which tends to diminish with diminishing intelligence.

Background

The van Hiele level theory of geometric thinking (Hoffer, 1983; van Hiele, 1986; Wirszup, 1976) contains important pedagogical/ psychological concepts and is philosophically elegant. Its context is the study of student thinking processes in Euclidean geometry where geometric knowledge is manifested through five qualitatively different, hierarchical (Perdikaris,2011b) and continuous levels (Perdikaris, 2011a). The student geometric development is not a monotonous increase of knowledge but an expanding equilibration (Confrey, 1994) and its aspects are argued by Battista (1994).

A capsule version of the levels as suggested by Burger and Shaughnessy (1986) is as follows:

Level 1 (Visualization). The student reasons about basic geometric concepts, such as simple shapes, primarely by means of visual considerations of the concept as a whole without explicit regard to properties of its components.

Level 2 (Analysis). The student reasons about geometric concepts by means of an informal analysis of component parts and attributes. Necessary properties of the concept are established.

Level 3 (Abstraction). The student logically orders the properties of concepts, forms abstract definitions, and can distinguish between the necessity and sufficiency of a set of properties in determining a concept.

Level .4 (Deduction). The student reasons formally within the context of a mathematical system, complete with undefined terms, an underlying logical system, definitions, and theorems.

Level 5 (Rigor). The student can compare systems based on different axioms and can study various geometries in the absence of concrete models.

This theory is based on the foundational ideas that all geometric knowledge rests on intuitive/exploratory experiences and thinking at a higher level is the result of thinking at lower levels. What is implicit at a level becomes explicit at the next level since, according to Freudenthal (1973), "the operational matter of the lower level bacomes a subject matter on the next level." The student learns to mathernatize, i.e., to organize the material by mathematical means (Perdikaris, 1996a). This van Hiele model of levels of geometric thinking has, according to Yazdani (2008), "implications not only for teaching and learning geometry but within other branches of mathematics and science as well."

Student geometric thinking during acquisition of the van Hiele levels deals with the meaning of information which involves ambiguity, i.e., verbally expressed aggregate uncertainty of a higher type than fuzziness (Foster, 2011; Klir and Wierman, 1998). Ambiguity is the condition where the meaning of information has several distinct possible interpretations and results from words that mean many different things, like the word "democracy."

A proper framework for analysis of the meaning of information that arises from thought processes and student cognition is possibility theory (Klir and Wierman, 1998; Zadeh,1978). This "soft" mathematical theory provides very natural and appropriate tools to model ambiguity since it is able to work in a pure qualitative way. It deals with the possible rather than the probable values of a variable with possibility being a matter of degree. Possibility can be identified by "compatibility" and linked with "difficulty." What is impossible must be improbable but whatever is possible need not be probable. This means that the degree of possibility always "'equals' or exceeds the degree of probability and a possibility distribution, which deals with the representation of meaning in natural languages, is not required to add up to 1.

Klir and Folger (1988, p. 191) state that the uncertainty is viewed as the capacity to acquire knowledge by a communicative act of some sort and Bertrand Russell, in Weber (1960, p. 229), refers to the capacity to acquire knowledge as intelligence. Thus, the ambiguity (aggregate possibilistic uncertainty) of student groups can be considered as the student group intelligence. The aim of this work is to compare the intelligence of student groups in a geometric task using a measure of ambiguity.

Methods and Procedures

The present work uses the data in the alternative paradigm for evaluating the acquisition of the van Hiele levels in three-dimensional geometry (Gutierrez, Jaime & Fortuny, 1991). The sample consisted of 50 students in three groups A, B and C with 20, 21 and 9 students respectively. All students had the same mathematical background. The authors tabulated the number of students attaining degrees of acquisition of each van Hiele level in Table 1. Notice that the first three van Hiele levels are considered since Level 4 rarely appears and Level 5 does not appear in secondary classrooms.

The work of Gutierrez et al.(l99l) is a creation of human mind and can be considered as an object. The term "object" may be defined as part of the world distinguishable as a single entity for an appreciable length of time. An observed characteristic of this object is the "acquisition of a level." This characteristic is an ordinal scaled characteristic since it is possible to distinguish its values by their intensities and to rank these intensities. It is also a discrete characteristic because it has a finite number of different values, i.e., acquisition of Level 1, acquisition of Level 2 and acquisition of Level 3.

These values, denoted by v_1 , v_2 and v_3, respectively, are fuzzy variables because each one can be expressed by five fuzzy sets (states), i.e., no acquisition, low acquisition, intermediate acquisition, high acquisition and complete acquisition, (Perdikaris, 2011a, Zadeh, 1965). The fuzzy sets above may be denoted by a, b, c, d and e respectively. Thus, there is a fuzzy system on the object because a set of interactive fuzzy variables is distinguished on it. This is an overall fuzzy system since it represents, as a whole, all the fuzzy variables involved. The student groups A, B and C are the backdrops, i.e., some sort of background against which the fuzzy variables are observed. The analysis below follows a method of mathematical psychology (Klir & Folger, 1988, pp. 282-285; Perdikaris, 2004).

For each student group (each observation) there are three 5-tuples of numbers, one 5-tuple for each fuzzy variable v_1, v_2 and v_3, and one number for each fuzzy set a, b, c, d and e (Table 2). The numbers are the membership degrees of the observed value of each fuzzy variable in the five fuzzy sets. For student group A, for example, the membership degree of the observed value of fuzzy variable v_3 (acquisition of Level 3) in the fuzzy set b (low acquisition) is 3/20. This, as it can be seen from Table 1, means that 3 of the 20 students of group A achieved low acquisition of Level 3.

Possibility distribution is used to characterize the constraint among the variables v_1, v_2 and v_3 of the overall fuzzy system. Membership degrees (pseudo-frequencies, i.e., frequencies that need not be whole numbers) of the overall states s (student profiles of acquisition of van Hiele levels) for student groups A, B and C are represented by m_A, m_B and m_C and the corresponding possibility distributions by r_A, r_B and r_C. All these are calculated using Table 2 and shown in Table 3. Notice that each possibility distribution is normalized, i.e., its values are divided by its maximum value so that the maximum value becomes 1 regardless of the number of overall states.

For each student group (each observation), the membership degree of an overall state s in Table 3 is calculated by taking the product of membership degrees of its components that are shown in Table 2. For student group A, fox example, the membership degree of the overall state s = (e d a) is m_A = (1)(6/20) (2/20) = 0.030 where 1 is the membership degree of the observed value of fuzzy variable v_1 (acquisition of Level 1) in the fuzzy set e (complete acquisition), 6/20 the membership degree of v_2 (acquisition of Level 2) in the fuzzy set d (high acquisition) and 2/20 the membership degree of v_3 (acquisition of Level 3) in the fuzzy set a (no acquisition).

For each student group (each observation), the degree of possibility of an overall state s is calculated by taking the quotient of its membership degree to the maximum membership degree of the overall states. For student group A, for example, the degree of possibility of the overall state s = (e d a) is r_A = 0.030/0.150 = 0.200 where 0.030 is the membership degree of the overall state s = (e d a) and 0.150 is the maximum membership degree of the overall states in m_A.

Nonspecificity (imprecision) is a type of uncertainty that emerges whenever some alternative belongs to a set of alternatives but it is not known which one in the set it is. Strife (conflict) is a type of uncertainty that expresses conflicting distinctions of the meaning of information. In the past, the sum of the measures of nonspecificity and strife has been used as a measure of ambiguity, i.e., aggregate possibilistic uncertainty (Perdikaris, 2002; Voskroglou, 2009). However, its justification is questionable since these measures do not satisfy all requirements that are considered essential on intuitive grounds (Klir and Wierman, 1998).

A well-justified measure of ambiguity that captures both nonspecificity and strife, the two types of uncertainty that coexist in possibility theory, is given by the function

$$AU(Pos) = -\sum_{i=1}^{n} p_i \log_2 p_i$$

which is a generalization of the classical Shannon entropy in possibility theory. Shis function is calculated by the Algorithm 3.2, using an ordered possibility distribution, $l=r_1 \geq r_2 \ldots \geq r_n$ and $r_{n+1} = 0$ (Klir and Wierman, 1998, pp. 98-100).

Consider, for example, the set X = {l,2, ... 12} and the normalized ordered possibility distribution r_C= <1, 0.750, 0.500, 0.500, 0.500, 0.370, 0.370, 0.250, 0.250, 0.245, 0.200, 0.125> in Table 3. The relevant values of $(r_j\text{-}r_{i+1})/(i+l\text{-}j)$ are listed in Table 4. For j=l and i = 3, for example, $(r_1\text{-}r_4)/(4\text{-}1)$ = (1-0.500)/3= 0.166. In the first pass (j = 1 and i = 1,2,..., 12) the maximum is reached at both i = 1 and i = 2. One takes the bigger value and puts $p_1 = p_2 = 0.250$. In second pass (j = 3 and i = 3,4,. ... ,12) the maximum is reached at both i = 7 and i = 12. One takes the bigger value and puts $p_3 = p_4 = p_5 = p_6 = p_7 = p_8 = p_9 = p_{10} = p_{11} = p_{12} = 0.050$. Then

$$AU(Pos) = -\sum_{i=1}^{n} p_i \log_2 p_i = 2(0.250)\log_2(0.250) + 10(0.050)\log_2(0.050) =$$
$$= 3.160$$

Similarly, the ambiguity (aggregate possibilistic uncertainty) of student groups A and B are 3.772 and 3.242 respectively.

Results and Conclusions

This work uses results from "soft" mathematics (fuzzy sets, possibilities) to measure the ambiguity (aggregate possibilistic uncertainty) of student groups in the context of van Hiele level theory of geometric thinking. Since ambiguity of student groups is seen as their intelligence a comparison of intelligence of student groups is possible. The student group A has the highest ambiguity and hence the highest intelligence. The same can be said for student group B in relation to student group C.

Intelligence, a primary aptitude of the mind, is the result of a number of independent abilities such as the capacity to comprehend and is widely used in educational settings (Perloff, Sternberg & Urbina, l996). It is a fuzzy (ill-defined) concept which can be developed by training in accordance with the "negatively accelerated" curve of learning. Intelligence can predict behavior and uniquely contributes to subsequent performance which is the competence on a certain task, i.e., the maximum potential for attainment of a goal in an achievement context. This performance tends to decrease with decreasing intelligence.

The overall fuzzy system above is a humanistic system in the context of van Hiele level theory and is characterized by the appearance of organized complexity, i.e., it has a moderate number of variables and shows the essential features of organization (Weaver, 1948). Systems of this type appear also in other developmental theories such as SOLO taxonomy (Biggs & Collis, 1982) and Bruner (1964a). Application of the measure AU (Pos) on these theories will probably establish it as a viable measure in educational research.

The methods used in this work are a qualitative innovation in mathematics education since they can handle, according to Zadeh (1978), "the intrinsic uncertainty of natural language which is a logical consequence of the necessity to express information in summarized form." The feasibility of using results from possibility theory rests on the fact that student behavior in geometric thinking involves uncertainty. This theory gives mathematical precision to concepts and thinking processes that are considered imprecise and extends them to more general contexts. It can be considered a methodology in mathematics education since it has a coherent collection of methods for the acquisition of new knowledge which can be used for the solution of a wide variety of problems. This methodology underlines in most cases the student abilities to think in approximate terms, and is tolerant to imprecision and conflict.

In this work one observes a relationship between the van Hiele level theory and a mathematical field of knowledge which offers keys of interpretation of the theory. Possibility theory facilitates understanding and operationalization of the theory which increases teachers' confidence and deepens their understanding of how students learn geometry. Thus, one acquires a new respect for the power of mathematical ideas in influencing pegagogical/psychological theories. However, the possibility theory, eventhough can be used as research tool in mathematics education, has not been given attention so far by the educational community.

Table 1
Number of Students Attaining Degrees of Acquisition of Each van Hiele Level (Adopted from Gutierrez et al, 1991)

		Degree of acquisition				
Group	van Hiele Level	No acquisition	Low	Intermediate	High	Complete
A	1	0	0	0	0	20
A	2	1	0	3	6	10
A	3	2	3	6	6	3
B	1	0	0	1	2	18
B	2	0	3	4	13	1
B	3	9	6	5	1	0
C	1	0	2	4	2	1
C	2	3	4	2	0	0
C	3	9	0	0	0	0

Table 2
Fuzzy Data of Variables v_1, v_2, v_3 Each With Fuzzy Sets a, b, c, d and e

			A	**B**	**C**
$v_1=$	{	**a**	0	0	0
		b	0	0	2/9
		c	0	1/21	4/9
		d	0	2/21	2/9
		e	1	18/21	1/9
$v_2=$	{	**a**	1/20	0	3/9
		b	0	3/21	4/9
		c	3/20	4/21	2/9
		d	6/20	13/21	0
		e	10/20	1/21	0
$v_3=$	{	**a**	2/20	9/21	1
		b	3/20	6/21	0
		c	6/20	5/21	0
		d	6/20	1/21	0
		e	3/20	0	0

Table 3
Possibility Distribution Estimates from Fuzzy Data.

v_1 v_2 v_3	m_A	r_A	m_B	r_B	m_C	r_C
s= e e e	0.075	0.500	0	0	0	0
e e a	0.050	0.333	0.017	0.075	0	0
e e b	0.075	0.500	0.012	0.053	0	0
e e c	0.150	1	0.010	0.044	0	0
e e d	0.150	1	0.002	0.009	0	0
e d a	0.030	0.200	0.227	1	0	0
e d b	0.045	0.300	0.150	0.660	0	0
e d c	0.090	0 . 60 0v	0.126	0.555	0	0
e a a	0.005	0.033	0	0	0.040	0.200
e b a	0	0	0.052	0.229	0.050	0.250
e e a	0.015	0.100	0.070	0.308	0.025	0.125
e c b	0.023	0.153	0.047	0.207	0	0
e b b	0	0	0.035	0.154		0
c a a	0	0	0	0	0.150	0.750
c b a	0	0	0.003	0.011	0.200	1
d a a	0	0	0	0	0.074	0.370
d b a	0	0	0.006	0.026	0.100	0.500
b a a	0	0	0	0	0.074	0.370
b b a	0		0	0	0.100	0.500
c c a	0	0	0.004	0.018	0.100	0.500
e c c	0.045	0.300	0.038	0.167	0	0
e c d	0.045	0.300	0:. 008 -	0.035	0	0
e d d	0.090	0.600	0.025	0.110	0	0
e d e	0.045	0.300	0	0	0	0
d d a	0	0	0.025	0.110	0	0
e b c	0	0	0.029	0.128	0	0
e a b	0.008	0.053	0	0	0	0
e c e	0.023	0.153	0	0	0	0
e a c	0.015	0.100	0	0	0	0
e a d	0.015	0.100	0	0	0	0
e a e	0.008	0.053	0	0	0-	0
d b b	0	0	0.004	0.018	0	0
d b c	0	0	0.003	0.013	0	0
d c a	0	0	0.008	0.035	0.049	0.245
d e b	0	0	0.005	0.022	0	0
d e c	0	0	0.004	0.018	0	0
d d b	0	0	0.017	0.075	0	0
d d c	0	0	0.0:14.	0.062	0	0
d d d	0	0	0.003	0.013	0	0
d e a	0	0	0.002	0.009	0	0
d e b	0	0	0.001	0.004	0	0

d e c	0	0	0.001	0.004	0	0
b c a	0	0	0	0	0.050	0.250
e b b	0	0	0.001	0.013	0	0
c b c	0	0	0.002	0.009	0	0
c c b	0	0	0.003	0.011	0	0
c c c	0	0	0.002	0.009	0	0
c d a	0	0	0.011	0.057	0	0
c d b	0	0	0.008	0.035	0	0-
c d c	0	0	0.007	0.031	0	0
c d d	0	0	0.001	0.004	0	0

Table 4

The Values of $(r_j-r_{i+1})/(i+1-j)$

	$(r_j-r_{i+1})/(i+1-j)$	
Pass	1	2
i\j	1	3
	0.250	
2	0.250	
3	0.166	0
4	0.125	0
5	0.126	0.043
6	0.105	0.033
7	0.107	0.050
8	0.094	0.042
9	0.084	0.036
10	0.080	0.038
11	0.080	0.042
12	0.083	0.050

† *Steve C. Perdikaris, Ph.D.*, Technological Educational Institute of Messologi, Messologi, Greece

References

Battista, M.T. (1994). On Greeno's Environmental/Model View of Conceptual Domains: A Spatial/Geometric Perspective. Journal for Research: in Mathematics Education, 25 (1) 86-99.

Biggs, J.B. and Collis, K.F. (1982). Evaluating the Quality of Learning. New York, NY: Academic" Press.

Bruner, J.S. (1964a). The Course of Cognitive Growth. American Psychologist, 19, 1-15.

Burger and Shaughnessy, J.M. (1986). Characterizing the van Hiele Levels of Development in Geometry. Journal for Research in Mathematics Education, 17(1), 31-48.

Confrey, J. (1994). A Theory of Intellectual Development. For the learning of Mathematics, 14(3),

Foster, C. (2011). Productive Ambiguity in the Learning of mathematics. For the Learning of Mathematics, 3l(2), 3-7.

Freudenthal, H. (1973). Mathematics as an Educational Task. Dordrecht: D. Reidel.

Gutierrez,A.,Jaime,A. & Fortuny,R.(1991). An Alternative Paradigm to Evaluate the Acquisition of the van Hiele Levels. Journal for Research in Mathematics Education, 22(3), 237-251.

Hoffer A.(1983). Van Hiele-Based Research. In R. Lesh and M. Landau (Eds.), Acquisition of Mathematics Concepts and Processes (pp. 205-227). New York, NY: Academic Press.

Klir, G.J. and Folger, T.A.(1988). Fuzzy Sets, Uncertainty and Information. London: Prentice-Hall International.

Klir, G.J. and Wierman, J.M.(1998) Uncertainty-Based Information. Heidelberg: Physica-Verlag.

Perdikaris, S.C. (1996a). Mathematizing the van Hiele Levels: A Fuzzy Approach. International Journal of Mathematical Education in Science and Technology, 27(1), 41-47.

Perdikaris, S.C. (1996b). A Systems Framework for Fuzzy Sets in the van Hiele Theory of Geometric Reasoning. International. Journal of Mathematical Education in Science and Technology, 27(2), 273-278.

Perdikaris, S.C. (2002). Measuring the Student Group Capacity for Obtaining Geometric Information in the van Hiele Developmental Thought Process, Fuzzy Systems and Mathematics, 16(3), China.

Perdikaris,S.C.(2004). The problem of Transition Across Levels in the van Hiele Theory of Geometric Reasoning. Philosophy of Mathematics Education Journal. Online: http://www.people.ex.ac.uk/PErnest.

Perdikaris, S.C. (2011a). Using Fuzzy Sets to Determine the Continuity of the van Hiele Levels. Journal of Mathematical Sciences and Mathematics Education, 5(1), 39-46.

Perdikaris, S.C. (2011b). Using the Cognitive Styles to Explain an Anomaly in the Hierarchy of the van Hiele Levels. Journal of Mathematical Sciences and Mathematics Education, 6(2), 35-43.

Perloff, R., Sternberg, R.J. and Urbina, S.(1996). Intelligence: Knowns and Unknowns. American Psychologist, 51.

van Hiele, P.M. (1986). Structure and Insight; A Theory of Mathematics Education. Orlando, Fla: Academic Press.

Voskoglou, M.G. (2009). Transition Across Levels in the Process of Learning: A Fuzzy Model. Journal of Mathematical Modeling and Application, 1(1), 37-44.

Weaver, W. (1948). Science and Complexity. American Scientist, 36, 536-544.

Weber, C.O. (1960). Basic Philosophies of Education. New York: Holt, Rinehart and Winston.

Wirszup, I. (l976). Breakthroughs in the Psychology of Learning and Teaching Geometry. In J.L.Martin and D.A. Bradbard (Eds.), Space and Geometry: Papers from a Research. Workshop (pp. 75-97). Columbus, OH: ERIC Center for Science, Mathematics and Environmental Education.

Yazdani, M.A.(2008). The Gagne-van Hiele Connection: A Comparative Analysis of Two Theoretical Learning Frameworks. Journal of Mathematical Sciences and Mathematics Education, 3(2), 58-63.

Zadeh, L.A.(l965). Fuzzy Sets. Information and Control, 8, 338-353.

Zadeh, L.A.(l978). Fuzzy Sets As a Basis for a Theory of Possibility. Fuzzy Sets and Systems, 1, 3-28.