A converse of the mean value theorem for integrals of functions of one or more variables

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Abstract

Let \( f \) be a continuous function of \( x \) on \( \Omega \), where \( \Omega \subseteq \mathbb{R}^N \), \( N \geq 1 \), is a bounded, open, convex, connected set. We prove that if \( f (\tilde{c}) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \), where \( \tilde{c} \in \Omega \) is a given point, then there exists a set \( S \subseteq \Omega \) such that \( f (\tilde{c}) = \frac{1}{|S|} \int_S f(x)dx \).

Introduction

One version of the Mean Value Theorem of integral calculus states that if \( f \) is a continuous function of \( x \) on a given compact, connected set \( V \subseteq \mathbb{R}^N \), then there exists a point \( \tilde{c} \in V \) such that \( \frac{1}{|V|} \int_V f(x)dx = f (\tilde{c}) \) (see, e.g., [2]).

The question to be considered here is: If \( f (\tilde{c}) \) is the value of a continuous function \( f \) at a given point \( \tilde{c} \in \Omega \), where \( \Omega \subseteq \mathbb{R}^N \), \( N \geq 1 \), is a bounded, open, convex, connected set, then does there exist a set \( S \subseteq \Omega \) such that \( f (\tilde{c}) = \frac{1}{|S|} \int_S f(x)dx \)?

In this paper, we prove that if \( f (\tilde{c}) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \), then there exists a set \( S \subseteq \Omega \) such that \( f (\tilde{c}) = \frac{1}{|S|} \int_S f(x)dx \).

In previous related work by other researchers, several papers have studied the converse of the Mean Value Theorem for functions of one variable. Tong and Braza [4] proved that given a continuous function \( f : [a, b] \rightarrow \mathbb{R} \) and given \( c \in (a, b) \) such that \( c \) is not an accumulation point of the set \( \{ x \in (a, b) : f (x) = f (c) \} \) and \( c \) is not a local extremum point of \( f \), then there exists \( (\alpha, \beta) \subseteq (a, b) \), where \( c \in (\alpha, \beta) \), such that \( \int_{\alpha}^{\beta} f(x)dx = f(c) (\beta - \alpha) \).

In related work on the Mean Value Theorem for differentiable functions \( F \) of one variable, Tong and Braza [5] and Mortici [3] proved that if \( F \) is continuous.
on \([a, b]\) and differentiable on \((a, b)\), then there exists an interval \((\alpha, \beta) \subset (a, b)\) such that \(F(\beta) - F(\alpha) = F'(c)(\beta - \alpha)\), provided \(F'\) satisfies certain hypotheses. These hypotheses are that either \(F'(c)\) is not a local extremum value of \(F'(x)\) on \((a, b)\) and \(c\) is not an accumulation point of the set \(\{x \in (a, b): F'(x) = F'(c)\}\), in which case \(c \in (\alpha, \beta)\), or alternatively that \(F'(c)\) is not a global extremum value of \(F'(x)\) on \((a, b)\), in which case \(c\) is not necessarily inside \((\alpha, \beta)\). Almeida [1] proved that if \(F\) is continuous on \([a, b]\) and differentiable on \((a, b)\), then there exists an interval \((\alpha, \beta) \subset (a, b)\) with \(c \in [\alpha, \beta]\) such that \(F(\beta) - F(\alpha) = F'(c)(\beta - \alpha)\), provided that there exists \(k_0 > 0\) such that 
\[ (c - k_0, c + k_0) \subset (a, b) \text{ and } F'(c - k) \leq F'(c) \leq F'(c + k) \text{ for all } k \in (0, k_0). \]

We have not seen work related to the converse of the Mean Value Theorem for integrals of functions of several variables.

A converse of the mean value theorem for integrals

We present the results of this paper in two theorems. The first theorem considers the existence of a set \(S \subset \Omega\) such that \(f(\bar{c}) = \frac{1}{|S|} \int_S f(\bar{x}) d\bar{x}\).

The second theorem concerns conditions under which \(\bar{c} \in S\) for the special case in which \(S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N]\).

We begin by proving the following theorem:

**Theorem 1:** Let \(f: \Omega \to R\) be a continuous function of \(\bar{x} \in \Omega\), where \(\Omega \subset \mathbb{R}^N, N \geq 1\), is a bounded, open, convex, connected set. Let \(f(\bar{c})\) be the value of \(f\) at a given point \(\bar{c} \in \Omega\).

If there exists an open set \(A \subset \Omega\) such that \(f(\bar{x}) = f(\bar{c})\) for all \(\bar{x} \in A\), then \(f(\bar{c}) = \int_A f(\bar{x}) d\bar{x}\).

If there does not exist an open set \(A \subset \Omega\) such that \(f(\bar{x}) = f(\bar{c})\) for all \(\bar{x} \in A\), then we have the following cases:
Case 1: If \( f(\bar{c}) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \), then there exists a set \( S_0 \subset \Omega \) such that
\[
f(\bar{c}) = \|S_0^{-1} \int_{S_0} f(\bar{x}) d\bar{x}.
\]

Case 2: If \( f(\bar{c}) \) is the absolute maximum value of \( f \) in \( \Omega \), then there exists a set \( S_1 \subset \Omega \) and a positive constant \( \varepsilon_1 \) such that
\[
f(\bar{c}) = \|S_1^{-1} \int_{S_1} f(\bar{x}) + \varepsilon_1 d\bar{x}.
\]

Case 3: If \( f(\bar{c}) \) is the absolute minimum value of \( f \) in \( \Omega \), then there exists a set \( S_2 \subset \Omega \) and a positive constant \( \varepsilon_2 \) such that
\[
f(\bar{c}) = \|S_2^{-1} \int_{S_2} f(\bar{x}) - \varepsilon_2 d\bar{x}.
\]

Proof:

If there exists an open set \( A \subset \Omega \) such that \( f(\bar{x}) = f(\bar{c}) \) for all \( \bar{x} \in A \), it immediately follows that \( f(\bar{c}) = \|A^{-1} \int_{A} f(\bar{x}) d\bar{x} \). Therefore, now suppose that there does not exist an open set \( A \subset \Omega \) such that \( f(\bar{x}) = f(\bar{c}) \) for all \( \bar{x} \in A \).

We have three possible cases to consider:

(1) Case 1 is the case in which \( f(\bar{c}) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \).

(2) Case 2 is the case in which \( f(\bar{c}) \) is the absolute maximum value of \( f \) in \( \Omega \).

(3) Case 3 is the case in which \( f(\bar{c}) \) is the absolute minimum value of \( f \) in \( \Omega \).

Note that we are not assuming that \( f \) has an absolute maximum value or absolute minimum value in \( \Omega \).

We now consider each case separately.

Case 1: Suppose that \( f(\bar{c}) \) is not the absolute (global) maximum or absolute (global) minimum value of \( f \) in \( \Omega \). Let \( g(\bar{x}) = f(\bar{x}) - f(\bar{c}) \). Then
$g\left(\bar{c}\right) = 0$. Since $f\left(\bar{c}\right)$ is not the absolute maximum value of $f$ in $\Omega$, there exists a point $\bar{x}_1 \in \Omega$ such that $g\left(\bar{x}_1\right) > 0$. Since $g\left(\bar{x}\right)$ is continuous, there exists an open ball $B_1 = B\left(\bar{x}_1, \partial_1\right)$ of radius $\partial_1$ about the point $\bar{x}_1$, such that $\overline{B}_1 \subset \Omega$ and such that $g\left(\bar{x}\right) > 0$ for $\bar{x} \in B_1$.

Since $f\left(\bar{c}\right)$ is not the absolute minimum value of $f$ in $\Omega$, there exists a point $\bar{x}_2 \in \Omega$ such that $g\left(\bar{x}_2\right) < 0$. Since $g\left(\bar{x}\right)$ is continuous, there exists an open ball $B_2 = B\left(\bar{x}_2, \partial_2\right)$ of radius $\partial_2$ about the point $\bar{x}_2$, such that $\overline{B}_2 \subset \Omega$ and such that $g\left(\bar{x}\right) < 0$ for $\bar{x} \in B_2$.

Since $\overline{B}_1 \subset \Omega$ and $\overline{B}_2 \subset \Omega$, and since $\Omega$ is a connected open set in $\mathbb{R}^N$, it follows that there exists a connected open set $U \subset \Omega$ such that $B_1 \subset U$, and such that $B_2 \subset U$, and such that the distance $\partial_2$ from the boundary of $U$ to the boundary of $\Omega$ is positive, so that $\overline{U} \subset \Omega$. Therefore $B_3 = B\left(\bar{x}, \partial_3\right) \subset \Omega$ for any $\bar{x} \in U$. Let $\partial_4 = \min\{\partial_1, \partial_2, \partial_3\}$. We now define

$$G\left(\bar{x}\right) = \frac{1}{|B(\bar{x}, \partial_4)|} \int_{B(\bar{x}, \partial_4)} g(y)dy,$$ where $\bar{x} \in U$.

It follows that $G$ is a continuous function of $\bar{x}$ on $U$, and $G\left(\bar{x}_2\right) < 0$ and $G\left(\bar{x}_1\right) > 0$, since $g\left(\bar{y}\right) < 0$ for $\bar{y} \in B\left(\bar{x}_2, \partial_4\right) \subset B_2$ and $g\left(\bar{y}\right) > 0$ for $\bar{y} \in B\left(\bar{x}_1, \partial_4\right) \subset B_1$.

Since $G$ is continuous on the connected set $U$, and $G\left(\bar{x}_2\right) < 0$ and $G\left(\bar{x}_1\right) > 0$, where $\bar{x}_1 \in U$ and where $\bar{x}_2 \in U$, then by the Intermediate Value Theorem (see, e.g., [2]) there exists a point $\bar{x}_3 \in U$ such that $G\left(\bar{x}_3\right) = 0$. Therefore

$$0 = G\left(\bar{x}_3\right) = \frac{1}{|B(\bar{x}_3, \partial_4)|} \int_{B(\bar{x}_3, \partial_4)} g(\bar{x})d\bar{x} = \frac{1}{|B(\bar{x}_3, \partial_4)|} \int_{B(\bar{x}_3, \partial_4)} f(\bar{c})d\bar{x}.$$ 

Re-arranging terms yields

$$f(\bar{c}) = \frac{1}{|B(\bar{x}_3, \partial_4)|} \int_{B(\bar{x}_3, \partial_4)} f(\bar{x})d\bar{x}.$$
We define $S_0 = B(\bar{x}_1, \partial_2)$ and the proof for Case 1 is complete.

**Case 2:** Suppose that $f(\bar{c})$ is the absolute maximum value of $f$ in $\Omega$. Let $B_0 = B(\bar{c}, \partial_0) \subset \Omega$ be the open ball of radius $\partial_0$ about the point $\bar{c}$ such that $\partial_0$ is the distance from $\bar{c}$ to the boundary of $\Omega$. Let $B_1 = B(\bar{c}, \partial_1)$ be the open ball of radius $\partial_1 < 1/2 \partial_0$ about the point $\bar{c}$. Note that $\overline{B_1} \subset \Omega$.

Let $g(\bar{x}) = f(\bar{x}) - f(\bar{c})$. Then $g(\bar{c}) = 0$, and $g(\bar{x}) \leq 0$ for $\bar{x} \in \Omega$. Let $\epsilon_1 = -\epsilon_0 \min_{\overline{B_1}} g(\bar{x})$, where $0 < \epsilon_0 < 1$. Note that $\epsilon_1 > 0$ (since otherwise it would follow that $\min_{x \in \overline{B_1}} g(\bar{x}) = 0 = \max_{x \in \overline{B_1}} g(\bar{x}) = g(\bar{c})$, which implies that $g(\bar{x}) = 0$ in $\overline{B_1}$ and so $f(\bar{x}) = f(\bar{c})$ on the set $A = B_1$, but this contradicts the assumption made at the start of the proof of this theorem that such an open set $A$ does not exist). Also note that $\epsilon_0$ can be arbitrarily small since $\epsilon_1 > 0$ can be arbitrarily small. And since $g$ is continuous on $\overline{B_1}$, it follows that there exists a point $\bar{x}_1 \in \overline{B_1}$ such that $g(\bar{x}_1) = \min_{x \in \overline{B_1}} g(\bar{x}) = -\epsilon_0 \epsilon_1$.

We have $-\epsilon_0 \epsilon_1 = \min_{x \in \overline{B_1}} g(\bar{x}) = g(\bar{x}_1) \leq g(\bar{x}) \leq g(\bar{c}) = \max_{x \in \overline{B_1}} g(\bar{x}) = 0$ for $\bar{x} \in \overline{B_1}$.

Now let $h(\bar{x}) = g(\bar{x}) + \epsilon_0$. It follows that $(1 - 1/\epsilon_0) \epsilon_1 = \min_{x \in \overline{B_1}} h(\bar{x}) = h(\bar{x}_1) \leq h(\bar{x}) \leq h(\bar{c}) = \max_{x \in \overline{B_1}} h(\bar{x}) = \epsilon_1$ for $\bar{x} \in \overline{B_1}$. And $(1 - 1/\epsilon_0) \epsilon_1 < 0$, since $0 < \epsilon_0 < 1$ and $\epsilon_1 > 0$.

Since $h$ is continuous on $\Omega$, and since $h(\bar{x}_1) < 0$ and $h(\bar{c}) > 0$, where $\bar{x}_1 \in \overline{B_1}$ and $\bar{c} \in \overline{B_1}$, it follows that exists a radius $\partial_2 < \partial_1$ such that $h(\bar{x}) < 0$ for $\bar{x} \in B(\bar{x}_1, \partial_2)$, and such that $h(\bar{x}) > 0$ for $\bar{x} \in B(\bar{c}, \partial_2)$.

We now define $H(\bar{x}) = \frac{1}{|B(\bar{x}, \partial_2)|} \int_{B(\bar{x}, \partial_2)} h(\bar{y}) d\bar{y}$, where $\bar{x} \in \overline{B_1}$. Note that
\( B(\bar{x}, \partial_2) \subset B(\bar{c}, \partial_0) \subset \Omega \) for \( \bar{x} \in \overline{B_1} \).

It follows that \( H \) is a continuous function of \( \bar{x} \) on \( \overline{B_1} \), and \( H(\bar{x}) < 0 \) and \( H(\bar{c}) > 0 \), since \( h(\bar{y}) < 0 \) for \( \bar{y} \in B(\bar{x}, \partial_2) \) and \( h(\bar{y}) > 0 \) for \( \bar{y} \in B(\bar{c}, \partial_2) \).

Since \( H \) is a continuous function of \( \bar{x} \) on the connected set \( \overline{B_1} \), and \( H(\bar{x}) < 0 \) and \( H(\bar{c}) > 0 \), then by the Intermediate Value Theorem there exists a point \( \bar{x}_2 \in \overline{B_1} \) such that \( H(\bar{x}_2) = 0 \). Therefore

\[
0 = \frac{1}{|B(\bar{x}_2, \partial_2)|} \int_{B(\bar{x}_2, \partial_2)} h(\bar{x})d\bar{x} = \frac{1}{|B(\bar{x}_2, \partial_2)|} \int_{B(\bar{x}_2, \partial_2)} f(\bar{x}) + \varepsilon_1 d\bar{x}.
\]

Re-arranging terms yields

\[
f(\bar{c}) = \frac{1}{|B(\bar{x}_2, \partial_2)|} \int_{B(\bar{x}_2, \partial_2)} f(\bar{x}) + \varepsilon_1 d\bar{x}.
\]

We define \( S_1 = B(\bar{x}_2, \partial_2) \) and the proof for Case 2 is complete.

**Case 3:**

Suppose that \( f(\bar{c}) \) is the absolute minimum value of \( f \) in \( \Omega \). Then \( f(\bar{x}) - f(\bar{c}) \geq 0 \) for \( \bar{x} \in \Omega \). Let \( v(\bar{x}) = -f(\bar{x}) \). And so \( v(\bar{x}) - v(\bar{c}) \leq 0 \) for \( \bar{x} \in \Omega \), and \( v(\bar{c}) \) is the absolute maximum value of \( v \) in \( \Omega \). From the proof of Case 2, it follows that there exists a point \( \bar{x}_3 \in \Omega \), and a radius \( \partial_3 \), and a positive constant \( \varepsilon_2 \) such that

\[
v(\bar{c}) = \frac{1}{|B(\bar{x}_3, \partial_3)|} \int_{B(\bar{x}_3, \partial_3)} v(\bar{x}) + \varepsilon_2 d\bar{x}.
\]

Since \( v(\bar{x}) = -f(\bar{x}) \), multiplying this equation by \(-1\) yields

\[
f(\bar{c}) = \frac{1}{|B(\bar{x}_3, \partial_3)|} \int_{B(\bar{x}_3, \partial_3)} f(\bar{x}) - \varepsilon_2 d\bar{x}.
\]

We define \( S_2 = B(\bar{x}_3, \partial_3) \) and the proof for Case 3 is complete.
This completes the proof of Theorem 1.

We now prove the following theorem:

**Theorem 2.** Let \( f : \Omega \rightarrow R \) be a continuous function of \( \bar{x} \in \Omega \), where \( \Omega \subset R^N, N \geq 1 \), is a bounded, open, convex, connected set. Let \( f(\bar{c}) \) be the value of \( f \) at a given point \( \bar{c} \in \Omega \).

If there exists an open set \( A \subset \Omega \) such that \( f(\bar{x}) = f(\bar{c}) \) for all \( \bar{x} \in A \), where \( \bar{c} \in A \), then
\[
f(\bar{c}) = \frac{1}{|A|} \int_A f(\bar{x})d\bar{x}.
\]

If there does not exist an open set \( A \subset \Omega \) such that \( f(\bar{x}) = f(\bar{c}) \) for all \( \bar{x} \in A \), where \( \bar{c} \in A \), then we have the following cases:

**Case 1:** Suppose the spatial dimension \( N=1 \).

If \( f(c) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \), then there exist \( a_1, b_1 \) in \( \Omega \) such that \( a_1 < c < b_1 \) and
\[
f(c) = \frac{1}{b_1-a_1} \int_{a_1}^{b_1} f(x)dx \quad \text{if and only if there exist } y_1, z_1 \text{ in } \Omega \text{ such that } y_1 < c < z_1 \text{ and } G(y_1), G(z_1) \text{ have the same sign or } G(y_1) = G(z_1) = 0,
\]
where
\[
G(t) = \int_c^t f(x) - f(c)dx.
\]

**Case 2:** Suppose the spatial dimension \( N \geq 2 \).

If \( f(\bar{c}) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \), then there exists a set \( S \subset \Omega \), where \( \bar{c} = (c_1, c_2, \ldots, c_N) \in S \), and where
\[
S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N],
\]
such that
\[
f(\bar{c}) = \frac{1}{|S|} \int_S f(\bar{x})d\bar{x}
\]
if for \( j=1,2,\ldots,N \), there exist \( y_j, z_j \) such that \( y_j < c_j < z_j \) and
\[
G_j(y_j), G_j(z_j) \text{ have the same sign or } G_j(y_j) = G_j(z_j) = 0,
\]
where
\[
G_j(t) = \int_y^t f(x) - f(c)dx.
\]
\[ G_j(t) = \int_{c_j}^{b_j} \int_{a_{j-1}}^{b_{j-1}} \ldots \int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_j, c_{j+1}, \ldots, c_N) - f(\tilde{c}) \, dx_1 \, dx_2 \ldots \, dx_j \]

where \( a_i, b_j \) are determined iteratively for each \( i \).

**Proof of Theorem 2:**

If there exists an open set \( A \subset \Omega \) such that \( f(x) = f(\tilde{c}) \) for all \( x \in A \), where \( \tilde{c} \in A \), it immediately follows that \( \frac{1}{|A|} \int_A f(\tilde{x}) \, d\tilde{x} \).

Therefore, now suppose that there does not exist an open set \( A \subset \Omega \) such that \( f(x) = f(\tilde{c}) \) for all \( x \in A \), where \( \tilde{c} \in A \).

We consider the cases in which the spatial dimension \( N=1 \) and in which \( N \geq 2 \) separately.

**Case 1:** First let \( N=1 \).

Suppose that \( f(c) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \). Let \( g(x) = f(x) - f(c) \). Then \( g(c) = 0 \), and \( g(c) \) is not the absolute maximum or absolute minimum value of \( g \) in \( \Omega \). We define

\[ G(t) = \int_t^1 g(x) \, dx = \int_t^1 f(x) - f(c) \, dx, \quad \text{where} \quad t \in \Omega = (a, b). \]

Note that \( G(c) = 0 \).

We begin by proving that there exist \( y_1, z_1 \) in \( \Omega \) such that \( y_1 < c < z_1 \) and \( G(y_1), G(z_1) \) have the same sign (i.e., both are positive or both are negative numbers) or \( G(y_1) = G(z_1) = 0 \) if and only if there exist \( a_1, b_1 \) in \( \Omega \) such that \( a_1 < c < b_1 \) and \( G(a_1) = G(b_1) \).

Therefore, suppose that there exist \( y_1, z_1 \) in \( \Omega \) such that \( y_1 < c < z_1 \) and \( G(y_1), G(z_1) \) have the same sign or \( G(y_1) = G(z_1) = 0 \). If

\[ G(y_1) = G(z_1) = 0 \]

then we are done. The desired result that \( G(a_1) = G(b_1) \) holds with \( a_1 = y_1 \) and \( b_1 = z_1 \).
Next, suppose that $G(y_1), G(z_1)$ have the same sign. If $G(y_1) = G(z_1)$ then we are done. The desired result that $G(a_i) = G(b_i)$ holds with $a_i = y_1$ and $b_i = z_1$.

Next, suppose that $G(y_1), G(z_1)$ have the same sign and $G(y_1) \neq G(z_1)$. First, assume that $0 < G(y_1) < G(z_1)$. Recall that $G(c) = 0$ and that $y_1 < c < z_1$. Therefore, by the continuity of $G(t)$ and the Intermediate Value Theorem, it follows that there exists $z_2$ in $\Omega$ such that $c < z_2 < z_1$ and $G(z_2) = G(y_1)$. The desired result that $G(a_i) = G(b_i)$ holds with $a_i = y_1$ and $b_i = z_2$.

Similarly, if $0 < G(z_1) < G(y_1)$, it follows that there exists $y_2$ in $\Omega$ such that $y_1 < y_2 < c$ and $G(y_2) = G(z_1)$. The desired result that $G(a_i) = G(b_i)$ holds with $a_i = y_2$ and $b_i = z_1$.

And if $G(z_1) < G(y_1) < 0$, it follows that there exists $z_3$ in $\Omega$ such that $c < z_3 < z_1$ and $G(z_3) = G(y_1)$. The desired result that $G(a_i) = G(b_i)$ holds with $a_i = y_1$ and $b_i = z_3$.

Finally, if that $G(y_1) < G(z_1) < 0$, it follows that there exists $y_3$ in $\Omega$ such that $y_1 < y_3 < c$ and $G(y_3) = G(z_1)$. The desired result that $G(a_i) = G(b_i)$ holds with $a_i = y_3$ and $b_i = z_1$.

Conversely, suppose that there exist $a_i, b_i$ in $\Omega$ such that $a_i < c < b_i$ and $G(a_i) = G(b_i)$. Then $G(a_i) = G(b_i) = 0$ or $G(a_i) = G(b_i) \neq 0$, in which case $G(a_i), G(b_i)$ have the same sign. Therefore, there exist $y_i, z_i$ in $\Omega$ such that $y_i < c < z_i$ and $G(y_i), G(z_i)$ have the same sign or $G(y_i) = G(z_i) = 0$, where we define $y_i = a_i$ and $z_i = b_i$.

Therefore, there exist $y_1, z_1$ in $\Omega$ such that $y_1 < c < z_1$ and $G(y_1), G(z_1)$ have the same sign or $G(y_1) = G(z_1) = 0$ if and only if there exist $a_i, b_i$ in $\Omega$ such that $a_i < c < b_i$ and $G(a_i) = G(b_i)$.
Since \( G(t) = \int_a^t g(x) \, dx = \int_a^t f(x) - f(c) \, dx \), it immediately follows that
\[
G(a_t) = G(b_t) \quad \text{if and only if} \quad f(c) = \frac{1}{b_t - a_t} \int_{a_t}^{b_t} f(x) \, dx.
\]

Therefore, there exist \( y_1, z_1 \) in \( \Omega \) such that \( y_1 < c < z_1 \) and
\[
G(y_1), G(z_1) \text{ have the same sign or } G(y_1) = G(z_1) = 0 \quad \text{if and only if there exist } a_1, b_1 \text{ in } \Omega \text{ such that } a_1 < c < b_1 \text{ and } f(c) = \frac{1}{b_1 - a_1} \int_{a_1}^{b_1} f(x) \, dx.
\]

This completes the proof of Case 1 of the theorem.

**Case 2:** Next, suppose \( N \geq 2 \).

Suppose that \( f(\bar{c}) \) is not the absolute maximum or absolute minimum value of \( f \) in \( \Omega \). Let \( g(x) = f(x) - f(\bar{c}) \). Then \( g(\bar{c}) = 0 \), and \( g(\bar{c}) \) is not the absolute maximum or absolute minimum value of \( g \) in \( \Omega \).

We next prove there exists a set \( S \subset \Omega \), where \( \bar{c} = (c_1, c_2, ..., c_N) \in S \) and where \( S = [a_1, b_1] \times [a_2, b_2] \times ... \times [a_N, b_N] \), such that
\[
f(\bar{c}) = \frac{1}{|S|} \int_S f(\bar{x}) \, d\bar{x} \quad \text{if for } j = 1, 2, ..., N, \text{ there exist } y_j, z_j \text{ such that } y_j < c_j < z_j \text{ and } G_j(y_j), G_j(z_j) \text{ have the same sign or } G_j(y_j) = G_j(z_j) = 0 \text{, where}
\]
\[
G_j(t) = \int_{c_j}^{y_j} \int_{a_{j-1}}^{b_{j-1}} \int_{a_{j-2}}^{b_{j-2}} ... \int_{a_{j-1}}^{b_{j-1}} g(x_1, x_2, ..., x_j, c_{j+1}, ..., c_N) \, dx_1 dx_2 ... dx_j.
\]

To prove this result, we will repeatedly apply the proof used in Case 1 for \( N=1 \).

We begin by defining \( G_1(t) = \int_{x_1}^{y_1} g(x_1, c_2, ..., c_N) \, dx_1 \) for \( t \) such that \( (t, c_2, ..., c_N) \in \Omega \). Recall that \( (c_1, c_2, ..., c_N) \in \Omega \). Also, recall that \( \Omega \) is a bounded, open, convex, connected set, so that \( (x_1, c_2, ..., c_N) \in \Omega \) on the interval of integration. And \( G_1(c_1) = 0 \).
By the proof from Case 1 for $N=1$, if there exist $y_1, z_1$ such that 

$$y_1 < c_1 < z_1$$

and $G_1(y_1), G_1(z_1)$ have the same sign or

$$G_1(y_1) = G_1(z_1) = 0,$$

then there exist $a_1, b_1$ such that $a_1 < c_1 < b_1$ and 

$$G_1(a_1) = G_1(b_1).$$

It immediately follows that 

$$0 = \int_{a_1}^{b_1} g(x_1, c_2, \ldots, c_N) \, dx_1.$$ 

Next, we define $G_j(t) = \int_{x_2}^{b_2} \int_{x_3}^{b_3} \cdots \int_{x_j}^{b_j} g(x_1, x_2, c_3, \ldots, c_N) \, dx_1 \, dx_2 \cdots dx_j$ for $t$ such that $(x_1, x_2, \ldots, x_j, c_j, \ldots, c_N) \in \Omega$ for $a_i \leq x_i \leq b_i$. Note that 

$$(x_1, x_2, \ldots, x_j, c_j, \ldots, c_N) \in \Omega$$

for $a_i \leq x_i \leq b_i$ by the previous step. Also, recall that $\Omega$ is a bounded, open, convex, connected set, so that $(x_1, x_2, c_3, \ldots, c_N) \in \Omega$ on the intervals of integration. And $G_j(c_j) = 0$.

By the proof from Case 1 for $N=1$, if there exist $y_2, z_2$ such that 

$$y_2 < c_2 < z_2$$

and $G_2(y_2), G_2(z_2)$ have the same sign or 

$$G_2(y_2) = G_2(z_2) = 0,$$

then there exist $a_2, b_2$ such that $a_2 < c_2 < b_2$ and 

$$G_2(a_2) = G_2(b_2).$$

It immediately follows that 

$$0 = \int_{a_2}^{b_2} \int_{a_3}^{b_3} \cdots \int_{a_j}^{b_j} g(x_1, x_2, c_3, \ldots, c_N) \, dx_1 \, dx_2 \cdots dx_j.$$ 

Next, for $j=3, \ldots, N$ we define 

$$G_j(t) = \int_{x_2}^{b_2} \int_{x_3}^{b_3} \cdots \int_{x_{j+1}}^{b_{j+1}} g(x_1, x_2, \ldots, x_{j+1}, c_{j+1}, \ldots, c_N) \, dx_1 \, dx_2 \cdots dx_j$$

for $t$ such that $(x_1, x_2, \ldots, x_{j+1}, c_{j+1}, \ldots, c_N) \in \Omega$ for $a_i \leq x_i \leq b_i$, $i=1, 2, \ldots, j-1$. Note that $(x_1, x_2, \ldots, x_{j+1}, c_{j+1}, \ldots, c_N) \in \Omega$ for $a_i \leq x_i \leq b_i$, $i=1, 2, \ldots, j-1$ by the previous steps. Also, recall that $\Omega$ is a bounded, open, convex, connected set, so that $(x_1, x_2, \ldots, x_{j+1}, c_{j+1}, \ldots, c_N) \in \Omega$ on the intervals of integration. And $G_j(c_j) = 0$.

By the proof from Case 1 for $N=1$, if there exist $y_j, z_j$ such that 

$$y_j < c_j < z_j$$

and $G_j(y_j), G_j(z_j)$ have the same sign or
\( G_j(y_j)=G_j(z_j)=0 \), then there exist \( a_j, b_j \) such that \( a_j < c_j < b_j \) and \( G_j(a_j)=G_j(b_j) \). It immediately follows that

\[
\int_{a_j}^{b_j} \int_{a_{j+1}}^{b_{j+1}} \ldots \int_{a_N}^{b_N} g(x_1, x_2, \ldots, x_j, c_{j+1}, \ldots, c_N) \, dx_1 \, dx_2 \ldots dx_j = 0.
\]

When \( j = N \), we obtain

\[
\int_{a_N}^{b_N} \int_{a_{N-1}}^{b_{N-1}} \ldots \int_{a_1}^{b_1} g(x_1, x_2, \ldots, x_N) \, dx_1 \, dx_2 \ldots dx_N = 0.
\]

Since \( g(\bar{x}) = f(\bar{x}) - f(\bar{c}) \), where \( \bar{x} = (x_1, x_2, \ldots, x_N) \), re-arranging terms in the above identity yields

\[
f(\bar{c}) = \frac{1}{\prod_{i=1}^{N} (b_i - a_i)} \int_{a_N}^{b_N} \int_{a_{N-1}}^{b_{N-1}} \ldots \int_{a_1}^{b_1} f(x_1, x_2, \ldots, x_N) \, dx_1 \, dx_2 \ldots dx_N.
\]

Therefore, the set \( S = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_N, b_N] \) satisfies

\[
f(\bar{c}) = \frac{1}{|S|} \int_S f(\bar{x}) \, d\bar{x} \quad \text{and} \quad \bar{c} = (c_1, c_2, \ldots, c_N) \in S.
\]

This completes the proof of Theorem 2.

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References


