The New Proof of Euler's Inequality Using Extouch Triangle

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Abstract

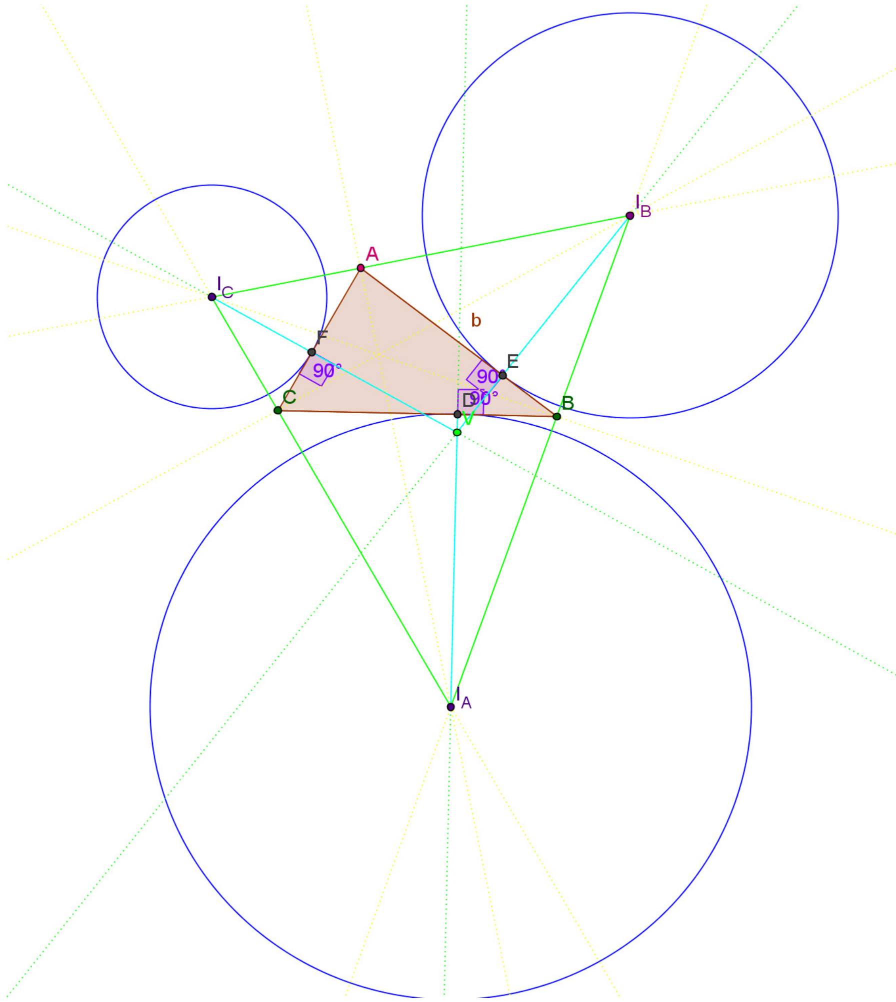
In this article we give a new proof of well-known Euler's Inequality using the properties of 'Ex-Touch triangle'.
Keywords: Euler's Inequality, Pedal's triangle, Ex-Touch triangle, Bevan point, Excentral triangle

Introduction

If R and r is the Circumradius and Inradius of a non-degenerate triangle then due to Euler we have an Inequality stated as $R \geq 2r$ and the equality holds when the triangle is equilateral. This ubiquitous inequality occurs in the literature in many different equivalent forms [4] and also Many other different simple approaches for proving this inequality are known. (some of them can be found in [2], [3], [5], [17], and [18]). In this article we present a proof for this Inequality based on two basic lemmas, one is on the fact "*Among all the pedal's triangles with respect to the points which are interior to the circumcircle of the reference triangle, the pedal triangle whose area is one fourth of area of the reference triangle has maximum area"* and other is related to the "*area of Ex-Touch triangle*".

Notations:

Let ABC be a triangle. We denote its side-lengths by a, b, c, its semiperimeter by $s = \dfrac{a+b+c}{2}$, its area by Δ, its circumradius by $R = \dfrac{abc}{4\Delta}$, and its inradius by $r = \dfrac{\Delta}{s}$, and if D, E, F are the points of contact of excircles opposite to the vertices A, B, C(A-excircle, B-excircle and C-excircle) of the $\Delta^{le} ABC$ with the sides BC, AC, AB then BD = AE = s-c, CD = AF = s-b, BF = CE = s-a respectively.



Formal definitions:

1. ***Pedal's Triangle:***

More generally Pedal's Triangle[6] with respect to P is the triangle formed by joining the foot of the perpendiculars drawn from an arbitrary point P which lies in the plane of triangle to the sides.

2. ***Euler's formula for area of pedal's triangle:***

If $\Delta^{|}$ is the area of pedal's triangle with respect to the point P and if S is the Circumcenter of $\Delta^{le}ABC$ then due to Euler we have

$$\Delta^{|} = \frac{\Delta}{4}\left|\left(1 - \frac{SP^2}{R^2}\right)\right|$$ [8], [12].

3. *Excentral triangle:*

If I_A, I_B, I_C are the excenters of the $\Delta^{le}ABC$ opposite to the vertices A, B, C respectively then the triangle formed by joining the points I_A, I_B, I_C is called as Excentral triangle. Clearly the $\Delta^{le}ABC$ is acts as orthic triangle(pedals triangle with respect to orthocenter) of Excentral triangle with respect to Incenter (I) of $\Delta^{le}ABC$ [1].

4. *Ex-Touch triangle:*

If D, E, F are the points of contact of excircles opposite to the vertices A, B, C with the sides BC, CA, AB respectively then the triangle formed by joining the points D, E, F is called as Extouch triangle[7]. It is well known that Ex-Touch triangle is a pedal's triangle with respect to Bevan point of $\Delta^{le}ABC$.

5. *Bevan point:*

Bevan point(V) [15] is the *Circumcenter* of the *Excentral triangle*. It is named in honor of ***Benjamin Bevan***. The Bevan point is the reflection of *Incenter* in the *Circumcenter* and it is also the reflection of *Orthocenter* in the *Spieker center*. The Bevan point(V) is the midpoint of line segment joining the *Nagel point* and *de-Longchamps point*.The Bevan point(V) is a triangle center and it is listed as the point **X(40)** in *Clark Kimberling's Encyclopedia of Triangle Centers***.**

Lemma – 1

If $\Delta^{|}$ is the area of pedal's triangle with respect to the any arbitrary point P, and if Δ is the area of reference triangle then prove that Among all the pedal's triangles, the pedal triangle with respect to some point P which is interior or on the boundary to the circumcircle of reference triangle and whose area is one fourth of the area of the reference triangle, has maximum area. That is $\frac{\Delta}{4} \geq \Delta^{|}$ for some P which lies interior or on the boundary to the circumcircle of reference triangle.

Proof: We know that by Euler's formula for the area of Pedal's triangle with respect to any point P which lies in the plane of triangle is given by

$$\Delta^{l} = \frac{\Delta}{4}\left|\left(1 - \frac{SP^2}{R^2}\right)\right|$$ where S is circumcenter of the reference triangle

CASE -1

When $1 - \frac{SP^2}{R^2} \geq 0 \Rightarrow SP \leq R$

That is the point P lies interior to the circumcircle of the triangle or the point P lies on the circumcircle of the triangle,

Now since $1 - \frac{SP^2}{R^2} \geq 0 \Rightarrow \left|\left(1 - \frac{SP^2}{R^2}\right)\right| = 1 - \frac{SP^2}{R^2}$

So $\Delta^{l} = \frac{\Delta}{4}\left|\left(1 - \frac{SP^2}{R^2}\right)\right| = \frac{\Delta}{4}\left(1 - \frac{SP^2}{R^2}\right)$

But we have $\frac{SP^2}{R^2} \geq 0 \Rightarrow \left(1 - \frac{SP^2}{R^2}\right) \leq 1 \Rightarrow \frac{\Delta}{4}\left(1 - \frac{SP^2}{R^2}\right) \leq \frac{\Delta}{4} \Rightarrow \Delta^{l} \leq \frac{\Delta}{4}$ ....…(1)

CASE -2

When $1 - \frac{SP^2}{R^2} < 0 \Rightarrow SP > R$

That is the point P lies exterior to the circumcircle of the triangle

Now since $1 - \frac{SP^2}{R^2} < 0 \Rightarrow \left|\left(1 - \frac{SP^2}{R^2}\right)\right| = \frac{SP^2}{R^2} - 1$

So $\Delta^{l} = \frac{\Delta}{4}\left|\left(1 - \frac{SP^2}{R^2}\right)\right| = \frac{\Delta}{4}\left(\frac{SP^2}{R^2} - 1\right)$

But we have $\frac{SP^2}{R^2} > 0 \Rightarrow \left(\frac{SP^2}{R^2} - 1\right) > -1 \Rightarrow \frac{\Delta}{4}\left(1 - \frac{SP^2}{R^2}\right) > -\frac{\Delta}{4} \Rightarrow \Delta^{|} > -\frac{\Delta}{4}$ …
…………(2)

From (1) and (2) it is clear that either $-\frac{\Delta}{4} < \Delta^{|}$ $or\ \Delta^{|} \le \frac{\Delta}{4}$ based on position of P

It implies $\frac{\Delta}{4} \ge \Delta^{|}$ when P lies interior or on the boundary of circumcircle of reference triangle.

NOTE:

(1). Clearly by Eulers formula for the area of pedals triangle we have

$$\Delta^{|} = \frac{\Delta}{4}\left|\left(1 - \frac{SP^2}{R^2}\right)\right|$$

, it clearly claims that the area of pedals triangle doesnot depend on the point P.

And The converse is also true. The locus of all points P in the plane of triangle ABC such that area of pedals triangle = $\Delta^{|}$ = k(constant) is defined by

$$\left|SP^2 - R^2\right| = \frac{4R^2k}{\Delta}$$

This is equivalent to $SP^2 = R^2 \pm \frac{4R^2k}{\Delta} = R^2\left(1 \pm \frac{4k}{\Delta}\right)$

If $k > \frac{\Delta}{4}$, then the locus is a circle of center S and radius $R_1 = R\sqrt{1 + \frac{4k}{\Delta}}$

If $k \le \frac{\Delta}{4}$, then the locus consists of two concentric circles of center S and radii $R\sqrt{1 + \frac{4k}{\Delta}}$ and $R\sqrt{1 - \frac{4k}{\Delta}}$, one of which degenerated to S when $k = \frac{\Delta}{4}$

Lemma – 2

If V is Bevan point and S is the circumcenter f any triangle then SV < R where R is circumradius.

Proof:

Let O, N and S are orthocenter, nine point center and circumcenter of reference triangle respectively.

we know that bevan point V is the circumcenter($S^{'}$) of excentral triangle,

and also we know that reference triangle acts as orthic triangle for the excentral triangle hence the Circumcenter(S) and Incenter(I) of reference triangle acts as nine point center($N^{'}$) and Orthocenter($O^{'}$) of excentral triangle respectively.

So by Euler's line property for the excentral triangle, Nine point center($N^{'}$) is the midpoint of the line joins orthocenter($O^{'}$) and circumcenter($S^{'}$) of excentral triangle.

Hence $O^{'}N^{'} = N^{'}S^{'}$, it implies that IS = SV

Since $IS < R \Rightarrow SV < R$

Hence proved

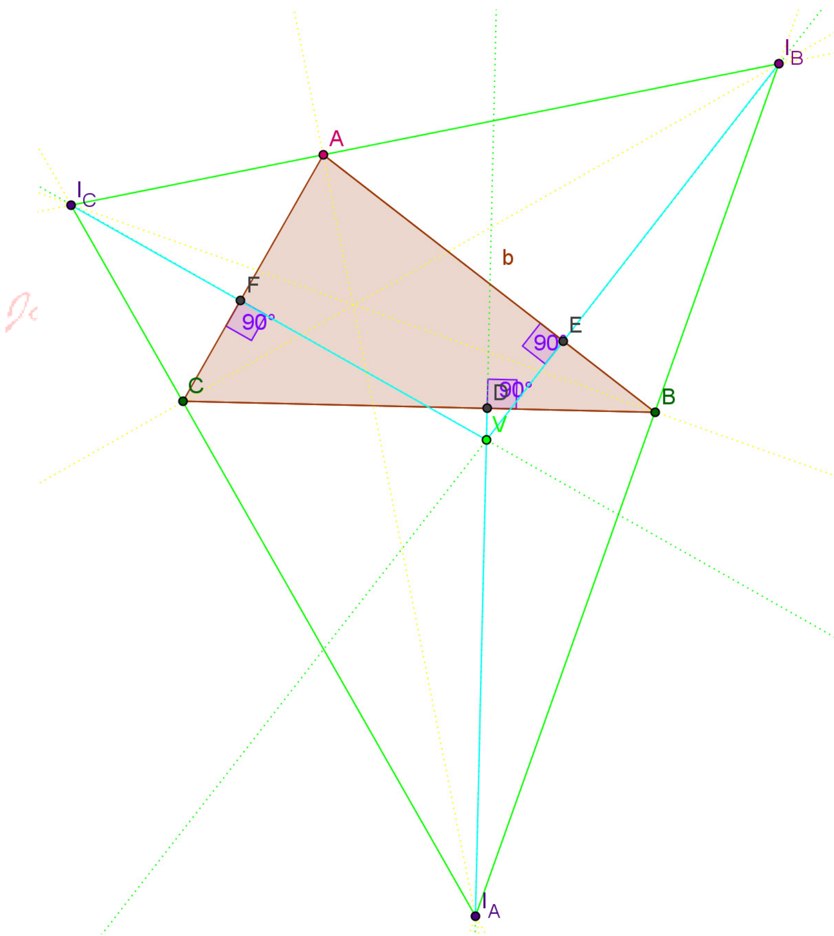
Lemma – 3

The Ex-Touch triangle of reference triangle acts as the pedal's triangle with respect to the Bevan point and the area of Ex-Touch triangle is $\dfrac{r\Delta}{2R}$

Proof:

Let D, E, F are the points of tangency of excircles opposite to the vertices A, B, C of $\Delta^{le}ABC$ with the sides BC, CA, AB and if I_A, I_B, I_C are the excenters opposite to the vertices A, B, C then I_AD, I_BE, I_CF are perpendicular to the sides BC, CA, AB respectively.

Let V is the point of intersection of extensions of I_AD, I_BE, I_CF respectively, now It is enough to prove that the point V is the circumcenter of Excentral triangle $I_AI_BI_C$ that is V is the Bevan point, it results that Ex-Touch triangle is the pedal's triangle of $\Delta^{le}ABC$ with respect to Bevan point[1].



Now by angle chasing and using little algebra we can prove that

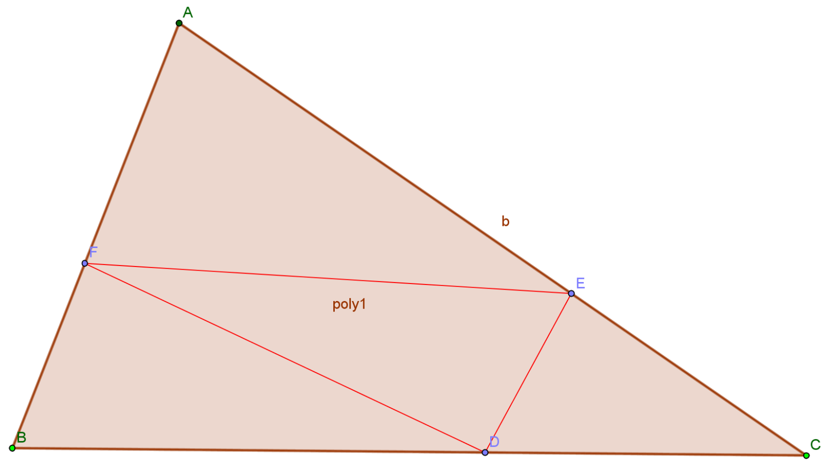
$\angle VI_A I_B = \angle VI_B I_A = \dfrac{C}{2}$, $\angle VI_C I_B = \angle VI_B I_C = \dfrac{A}{2}$ and

$\angle VI_C I_A = \angle VI_A I_C = \dfrac{B}{2}$

It implies that $I_AV = I_BV = I_CV$

So V is the Circumcenter of Excentral triangle $I_AI_BI_C$. Hence *Ex-Touch triangle is the pedal's triangle of* $\Delta^{le} ABC$ *with respect to Bevan point.*

Now let us find the area of Ex-Touch triangle,



We are familiar with the result that "In any triangle cevian divides the triangle into two triangles whose ratio between the areas is equal to the ratio between corresponding bases".

So $\dfrac{[\Delta BEC]}{[\Delta ABC]}=\dfrac{CE}{AC}$ and $\dfrac{[\Delta DEC]}{[\Delta BEC]}=\dfrac{CD}{CB}$ it implies that

$$[\Delta DEC]=\frac{(s-a)(s-b)\Delta}{ab}=\Delta\,\mathrm{Sin}^2\frac{C}{2}$$

In the same way $[\Delta EFA]=\dfrac{(s-b)(s-c)\Delta}{bc}=\Delta\,\mathrm{Sin}^2\dfrac{A}{2}$ and

$$[\Delta FDB]=\frac{(s-a)(s-c)\Delta}{ac}=\Delta\,\mathrm{Sin}^2\frac{B}{2}$$

Now $[\Delta DEF]=[\Delta ABC]-([\Delta AEF]+[\Delta BDF]+[\Delta CDE])$

By using the identity $\mathrm{Sin}^2\dfrac{A}{2}+\mathrm{Sin}^2\dfrac{B}{2}+\mathrm{Sin}^2\dfrac{C}{2}=1-\dfrac{r}{2R}$

We can arrive at a conclusion, $[\Delta DEF]=\dfrac{r\Delta}{2R}$

Now let us prove the Euler's Inequality

EULER'S INEQUALITY

If R is the Circumradius and r is the Inradius of triangle ABC then $R\geq 2r$ *and the equality holds when the triangle is equilateral.*

Proof:

Using lemma-1, lemma -2 and lemma -3 we have $\frac{\Delta}{4} \geq \Delta'$ and $\Delta' = \frac{r\Delta}{2R}$ since $SV < R$

It follows our desired Inequality, $R \geq 2r$.

Remark:

Clearly the Intouch triangle(Intouch triangle is the triangle formed by joining the points of contacts of incircle with the sides of reference triangle) is also pedals triangle with respect to Incenter(I) of reference triangle, and area of Intouch triangle is also $\frac{r\Delta}{2R}$ and also IS < R

So what ever proof we adopted to show euler's inequality using extouch triangle the same proof we can adopt to show euler's inequality using intouch triangle also.

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