Review on Generalized Pearson System of Probability Distributions

M. Shakil, Ph.D. †
B.M. Golam Kibria, Ph.D. ‡
J. N. Singh, Ph.D. §

Abstract

In recent years, many researchers have considered a generalization of the Pearson system, known as generalized Pearson system of probability distributions. In this paper, we have reviewed these new classes of continuous probability distribution which can be generated from the generalized Pearson system of differential equation. We have identified as many as 15 such distributions. It is hoped that the proposed attempt will be helpful in designing a new approach of unifying different families of distributions based on the generalized Pearson differential equation, including the estimation of the parameters and inferences about the parameters.

1. Introduction: Pearson System of Distributions

A continuous distribution belongs to the Pearson system if its pdf (probability density function) f satisfies a differential equation of the form

$$\frac{1}{f(x)}\frac{df_X(x)}{dx} = -\frac{x+a}{bx^2+cx+d}, \tag{1}$$

where a, b, c, and d are real parameters such that f is a pdf. The shapes of the pdf depend on the values of these parameters, based on which Pearson (1895, 1901) classified these distributions into a number of types known as Pearson Types I – VI. Later in another paper, Pearson (1916) defined more special cases and subtypes known as Pearson Types VII - XII. Many well-known distributions are special cases of Pearson Type distributions which include Normal and Student's t distributions (Pearson Type VII), Beta distribution (Pearson Type I), Gamma distribution (Pearson Type III) among others. For details on these Pearson system of continuous probability distributions, the interested readers are referred to Johnson et al. (1994).

2. Generalized Pearson System of Distributions

A continuous distribution belongs to the Pearson system if its pdf (probability density function) f satisfies a differential equation of the form

$$\frac{1}{f(x)}\frac{df_X(x)}{dx} = -\frac{x+a}{bx^2+cx+d}, \qquad (2)$$

where a, b, c, and d are real parameters such that f is a pdf. The shapes of the pdf depend on the values of these parameters, based on which Pearson (1895, 1901) classified these distributions into a number of types known as Pearson Types I – VI. Later in another paper, Pearson (1916) defined more special cases and subtypes known as Pearson Types VII - XII. Many well-known distributions are special cases of Pearson Type distributions which include Normal and Student's t distributions (Pearson Type VII), Beta distribution (Pearson Type I), Gamma distribution (Pearson Type III) among others. For details on these Pearson system of continuous probability distributions, the interested readers are referred to Johnson et al. (1994). In recent years, many researchers have considered a generalization of (2), known as generalized Pearson system of differential equation (GPE), given by

$$\frac{1}{f(x)}\frac{df_X(x)}{dx} = \frac{\sum_{j=0}^{m} a_j x^j}{\sum_{j=0}^{n} b_j x^j}, \qquad (2)$$

where m, $n \in \mathrm{N} / \{0\}$ and the coefficients a_j and b_j are real parameters. The system of continuous univariate $pdf\,'s$ generated by GPE is called a generalized Pearson system which includes a vast majority of continuous $pdf\,'s$ by proper choices of these parameters. We have identified as many as 14 such distributions, which are provided below:

a) Roy (1971) studied GPE, when $m = 2, n = 3, b_0 = 0$, to derive five frequency curves whose parameters depend on the first seven population moments.

b) Dunning and Hanson (1977) used GPE in his paper on generalized Pearson distributions and nonlinear programming.

c) Cobb et al. (1983) extended Pearson's class of distributions to generate multimodal distributions by taking the polynomial in the numerator of GPE of degree higher than one and the denominator, say $v(x)$, having one of the following forms:

(I) $v(x) = 1, \quad -\infty < x < \infty,$

(II) $v(x) = x, \quad 0 < x < \infty,$

(III) $v(x) = x^2, \quad 0 < x < \infty,$

(IV) $v(x) = x(1-x), \quad 0 < x < 1.$

d) Chaudhry and Ahmad (1993) studied another class of generalized Pearson distributions when

$$m = 4, n = 3, b_0 = b_1 = b_2 = 0, \frac{a_4}{2b_3} = -2\alpha, \frac{a_0}{2b_3} = 2\beta, b_3 \neq 0.$$

e) Lefevre et al. (2002) studied characterization problems based on some generalized Pearson distributions.

f) Considering the following class of GPE

$$\frac{1}{f(x)} \frac{df_X(x)}{dx} = \frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2},$$

Sankaran et al. (2003) proposed a new class of probability distributions and established some characterization results based on a relationship between the failure rate and the conditional moments.

g) Stavroyiannis et al. (2007) studied generalized Pearson distributions in the context of the superstatistics with non-linear forces and various distributions.

h) Rossani and Scarfone (2009) have studied GPE in the following form

$$\frac{1}{f(x)} \frac{df_X(x)}{dx} = -\frac{a_0 + a_1 x + a_2 x^2}{b_0 + b_1 x + b_2 x^2},$$

and used it to generate generalized Pearson distributions in order to study charged particles interacting with an electric and/or a magnetic field.

3. Some Recently Developed Generalized Pearson System of Distributions

In what follows, we provide a brief description of some new classes of distributions generated as the solutions of the generalized Pearson system of differential equation (GPE) (2).

Shakil et al (2010a) defined a new class of generalized Pearson distributions based on the following differential equation

$$\frac{df_X(x)}{dx} = \left(\frac{a_0 + a_1 x + a_2 x^2}{b_1 x} \right) f_X(x), \quad b_1 \neq 0, \qquad (3)$$

which is a special case of the GPE (2) when $m = 2, n = 1$, and $b_0 = 0$. The solution to the differential equation (3) is given by

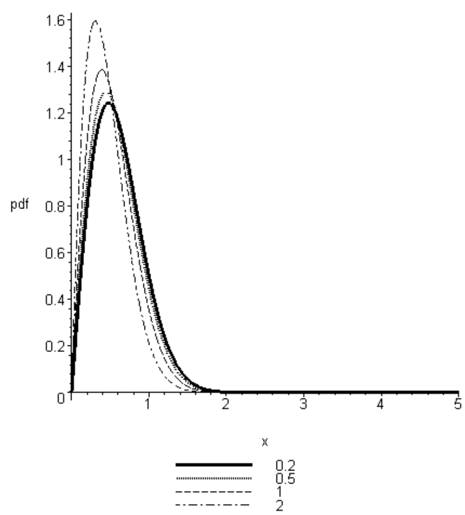
$$f_X(x) = C\, x^{\alpha} \exp\left(-\mu x^2 - \beta x\right), \quad \alpha \geq 0, \beta \geq 0, \mu > 0, x \geq 0, \qquad (4)$$

where $\mu = -\frac{a_2}{2b_1}$, $\alpha = \frac{a_0}{b_1}$, $\beta = -\frac{a_1}{b_1}$, $b_1 \neq 0$, and C is the normalizing constant given by

$$C = \frac{(2\mu)^{(\alpha+1)/2}}{\Gamma(\alpha+1)\, \exp\left(\beta^2 / (8\mu)\right) D_{-(\alpha+1)}\left(\beta / \left(\sqrt{2\mu}\right)\right)}, \qquad (5)$$

where $D_p(z)$ denotes the parabolic cylinder function. The possible shapes of the *pdf* f (4) are provided for some selected values of the parameters in Figure 1. It is clear from Figure 1 that the distributions of the random variable X are positively (that is, right) skewed and unimodal.

(a)



(b)

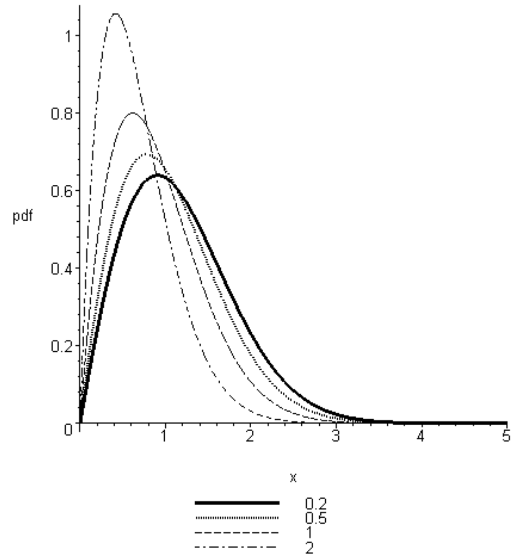


Figure 1: PDF Plots of X for (a) $\alpha = 1$, $\sigma = 0.5$, $\beta = 0.2, 0.5, 1, 2$ (left), and (b) $\alpha = 1$, $\sigma = 1$, $\beta = 0.2, 0.5, 1, 2$ (right).

Shakil and Kibria (2010) consider the GPE (2) in the following form

$$\frac{df_X(x)}{dx} = \left(\frac{a_0 + a_p x^p}{b_1 x + b_{p+1} x^{p+1}} \right) f_X(x), \quad b_1 \neq 0, b_{p+1} \neq 0, x > 0, \quad (6)$$

when $m = p$, $n = p + 1$, $a_1 = a_2 = \cdots = a_{p-1} = 0$, and $b_0 = b_2 = \cdots = b_p = 0$. The solution to the differential equation (6) is given by

$$f_X(x) = C x^{\mu - 1} \left(\alpha + \beta x^p \right)^{-\nu}, \; x > 0, \alpha > 0, \beta > 0, \mu > 0, \nu > 0, and\, p > 0 \quad (7)$$

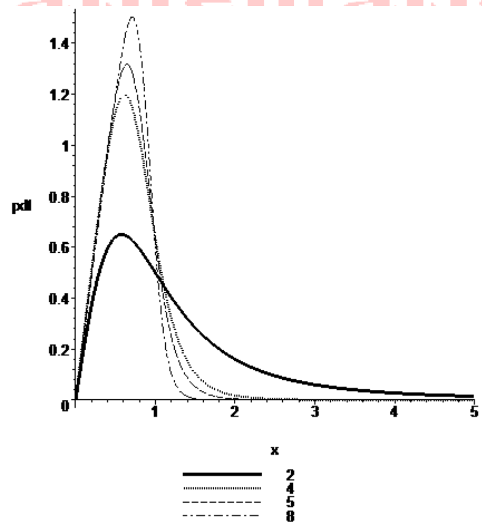
where $\alpha = b_1$, $\beta = b_{p+1}$, $\mu = \dfrac{a_0 + b_1}{b_1}$, $\nu = \dfrac{a_0 b_{p+1} - a_p b_1}{p b_1 b_{p+1}}$, $b_1 \neq 0, b_{p+1} \neq 0$,

and C is the normalizing constant given by

$$C = \frac{p\,(\alpha)^{\nu - \frac{\mu}{p}}\,(\beta)^{\frac{\mu}{p}}}{\mathrm{B}\left(\frac{\mu}{p}, \nu - \frac{\mu}{p}\right)}, \tag{8}$$

where $\mathrm{B}(.\,,.)$ denotes the beta function. By definition of beta function, the parameters in (8) should be chosen such that $\nu > \frac{\mu}{p}$. The possible shapes of the pdf f (7) are provided for some selected values of the parameters in Figure 2 (a, b) below. From these graphs, it is evident that the distribution of the RV X is right skewed.

(a)



(b)

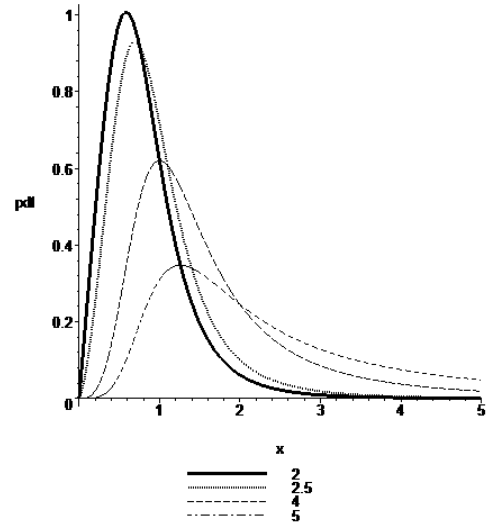


Figure 2: PDF Plots of X for (a)
$\alpha = 1, \beta = 1, \nu = 2, \mu = 2, p = 2, 4, 5, 8$ **(left); and (b)**
$\alpha = 1, \beta = 1, \nu = 2, p = 3, \mu = 2, 2.5, 4, 5$ **(right).**

Shakil, Kibria and Singh (2010b) consider the GPE (2) in the following form

$$\frac{df_X(x)}{dx} = \left(\frac{a_0 + a_p x^p + a_{2p} x^{2p}}{b_{p+1} x^{p+1}} \right) f_X(x), \quad b_{p+1} \neq 0, x > 0, \qquad (9)$$

Where: $m = 2p, n = p + 1, a_1 = a_2 = \cdots = a_{p-1} = a_{p+1} = \cdots = a_{2p-1} = 0,$ and $b_0 = b_1 = b_2 = \cdots = b_p = 0$. The solution to the differential equation (9) is given by

, (10)

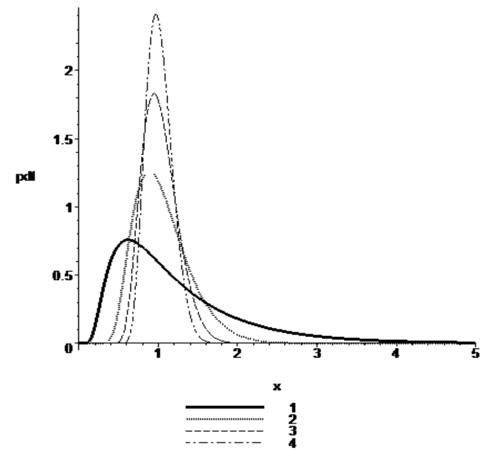
where $\alpha = -\dfrac{a_{2p}}{p b_{p+1}}, \ \beta = \dfrac{a_0}{p b_{p+1}}, \nu = \dfrac{a_p + b_{p+1}}{b_{p+1}}, b_{p+1} \neq 0, \ p > 0$, and C is the normalizing constant given by

$$C = \frac{p}{2}\left(\frac{\alpha}{\beta}\right)^{\frac{\nu}{2p}} \frac{1}{K_{\frac{\nu}{p}}\left(2\sqrt{\alpha\beta}\right)}, \tag{11}$$

where $K_{\frac{\nu}{p}}\left(2\sqrt{\alpha\beta}\right)$ denotes the modified Bessel function of third kind.

The possible shapes of the pdf f (10) are provided for some selected values of the parameters in Figure 3 (a, b). It is clear from Figure 3 (a, b), the distributions of the random variable X are positively (that is, right) skewed with longer and heavier right tails.

(a)



(b)

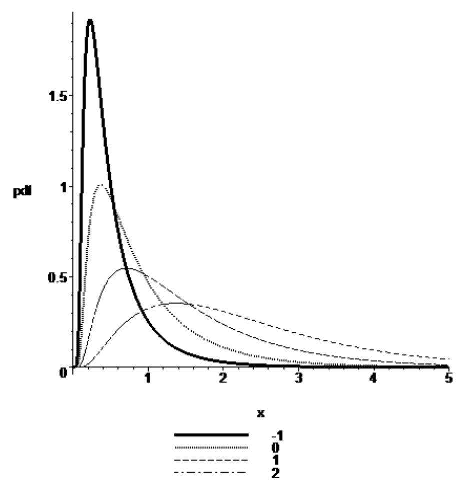


Figure 3: PDF Plots of X for (a) $\alpha = 1,\ \beta = 1,\ \nu = 0,\ p = 1, 2, 3, 4$ (left), and (b) $\alpha = 1,\ \beta = 0.5,\ p = 1,\ \nu = -1, 0, 1, 2$ (right).

4. Hamedani's Generalized Pearson System of Distributions

Hamedani (2011) has defined a new variation of SKS continuous probability distribution given in (10) in a bounded domain. The pdf of this distribution is given by

$$f(x) = C\,px^{-(p+1)}\left(\beta - \alpha x^{2p}\right)\exp\left(-\alpha x^{p} - \beta x^{-p}\right), \quad 0 < x < \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2p}}, \qquad (12)$$

where $\alpha > 0$, $\beta > 0$, and $p > 0$ are parameters and $C = \exp\left\{2\sqrt{\alpha\beta}\right\}$ is the normalizing constant. The cdf corresponding to the pdf (12) is given by

$$F(x) = C\exp\left(-\alpha x^{p} - \beta x^{-p}\right), \quad 0 < x < \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2p}}. \qquad (13)$$

For the special case of $\alpha = \beta$, we have

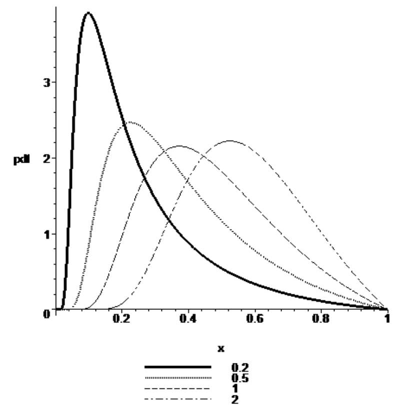
$$f(x) = \alpha\, p\, e^{2\alpha}\, x^{-(p+1)}\left(1 - x^{2p}\right)\exp\left(-\alpha(x^{p} + x^{-p})\right), \quad 0 < x < \left(\frac{\beta}{\alpha}\right)^{\frac{1}{2p}}, \qquad (14)$$

where $\alpha > 0$ and $p > 0$ are parameters. It is easy to see that the pdf f given by (12) satisfies the following differential equation

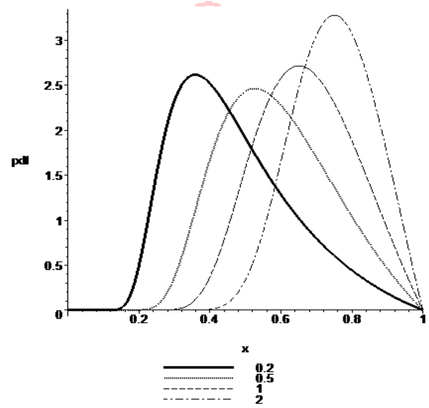
$$\frac{1}{f(x)}\frac{df(x)}{d\,x} = \frac{\beta^{2} p - \beta(p+1)x^{p} - 2\alpha\beta\, p\, x^{2p} - \alpha(p-1)x^{3p} + \alpha^{2} p x^{4p}}{\beta x^{p+1} - \alpha x^{3p+1}},$$

which is a special case of GPE (2). For characterizations of the pdf (12) when $p \in \mathrm{N} / \{0\}$, the interested readers are referred to Hamedani (2011). For the special case of $\alpha = \beta$, the possible shapes of the pdf f (14) are provided for some selected values of the parameters $\alpha = \beta = 0.2, 0.5, 1, 2$ for $p = 1, 2, 5$, in the following Figure 4 (a, b, c). The effects of the parameters can easily be seen from these graphs. For example, it is clear from the plotted Figure 4 (a, b, c) of the pdf that the newly proposed probability density function is unimodal. Also, for some selected values of the parameters, the distributions of the random variable X are both right and left skewed, whereas, for (i) $\alpha = \beta = 2$, $p = 1$, (ii) $\alpha = \beta = 0.5$, $p = 2$, and (iii) $\alpha = \beta = 0.2$, $p = 5$, the distributions appear to be symmetric.

(a)



(b)



(c)

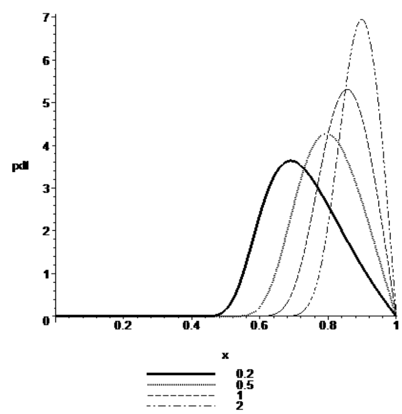


Figure 4: PDF for $\alpha = \beta = 0.2, 0.5, 1, 2$, for $p = 1, 2, 5$, respectively.

5. Ahsanullah, Shakil and Kibria's Generalized Pearson System of Distributions

Ahsanullah, Shakil and Kibria (2013) defined a new class of distributions as solutions of the GPE (2). They considered the following differential equation

$$\frac{df_X(x)}{dx} = \left(\frac{a_1 + a_2 x + a_3 x^2}{b_3 x^2 + b_4 x^3}\right) f_X(x),$$

(15)

which is a special case of the generalized Pearson Eq. (2) when $m = 2, n = 3$. Putting $b_3 = 1$, $b_4 = \gamma$, $a_1 = \beta\gamma$, $a_2 = \beta - \gamma + \gamma\nu$, $a_3 = \nu + \mu - 2$, $x > 0$; in (3), we have

$$\frac{1}{f_X(x)} \frac{df_X(x)}{dx} = \frac{\beta\gamma + (\beta - \gamma + \gamma\nu)x + (\mu + \nu - 2)x^2}{x^3 + \gamma x^2} = \frac{\nu - 1}{x} + \frac{\mu - 1}{x + \gamma} + \frac{\beta}{x^2},$$

where we assume that $\beta > 0$, $\gamma > 0$, $0 < \nu < 1$, $0 < \mu < 1$, $1 - \mu > \nu > 0$.

Integrating the above equation, we have

$$f_X(x) = C\, x^{\nu - 1} (x + \gamma)^{(\mu - 1)} \exp\left(-\beta x^{-1}\right), \quad 0 < x < \infty, \quad (16)$$

Using the equation (3.471.7), Page 340 of Gradshteyn and Ryzhik (1980), we easily obtain the following normalizing constant as

$$\frac{1}{C} = \beta^{(\nu-1)/2} \gamma^{\frac{\nu-1}{2}+\mu} \Gamma(1 - \mu - \nu) \exp\left(\frac{\beta}{2\gamma}\right) W_{\frac{\nu-1}{2}+\mu, \frac{-\nu}{2}}\left(\frac{\beta}{\gamma}\right), \qquad (17)$$

where W(.) denotes the Whittaker function which is defined as the solution of the following differential equation

$$\frac{d^2W}{dx^2} + \left(-\frac{1}{4} + \frac{\mu}{x} + \frac{1/4 - \mu^2}{x^2}\right) W = 0,$$

(See, for details, Abramowitz Milton and Stegun, Irene A. eds. Handbook of Mathematical Functions, chapter 13, Dover publications, New York, 1970, page 505). The possible shapes of the *pdf* *f(x)* as given in (16) are provided for some selected values of the parameters in the following Figure 5 (a, b). It is clear from Figure 5 (a, b), that the newly proposed distribution is right skewed and the effects of the parameters can easily be seen from these graphs.

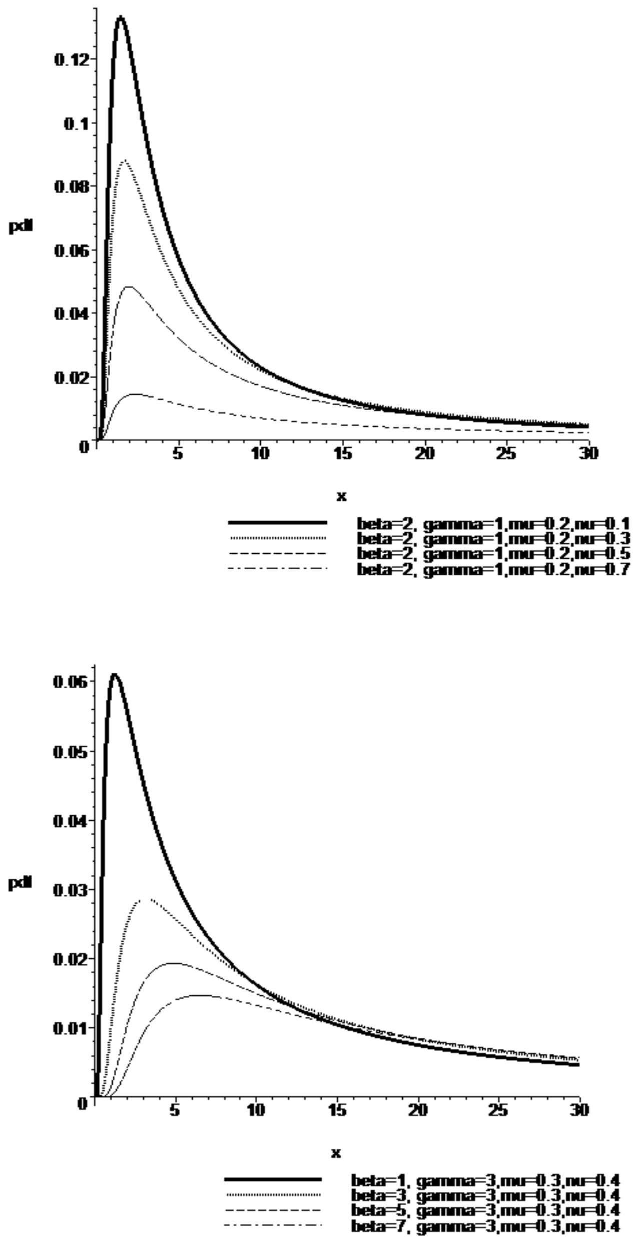


Figure 5(a): PDF for $\nu = 0.1, 0.3, 0.5, 0.7$ when $\beta = 2, \gamma = 1, \mu = 0.2$ Figure 5(b): PDF for $\beta = 1, 3, 5, 7$ when $\gamma = 3, \mu = 0.3, \nu = 0.4$

6. Stavroyiannis' Generalized Pearson System of Distributions

Recently, Stavroyiannis (2014) defined a new class of distributions as solutions of the GPE (2). They considered the following differential equation

$$\frac{1}{f(x)}\frac{df_X(x)}{dx} = \frac{\sum_{j=0}^{5} a_j x^j}{\sum_{j=0}^{6} b_j x^j}, \tag{18}$$

which is a special case of the generalized Pearson Eq. (2) when $m = 5$, $n = 6$. By taking special values of the coefficients a_j and b_j, Stavroyiannis (2014) obtained the GPE in the following form

$$\frac{1}{f(x)}\frac{df(x)}{d\,x} = -\frac{a^5\nu + 2a^4 m(x-\lambda) + a^3\nu(x-\lambda)^2 + 2a^2(1+2b)m(x-\lambda)^3 + ab\nu(x-\lambda)^4 + 4bm(x-\lambda)^5}{\left(a^2+(x-\lambda)^2\right)\left(a^4 + a^2(x-\lambda)^2 + b(x-\lambda)^4\right)}$$

with its solution given by the following probability density function:

$$f(x) = C\frac{1}{\left(1+\left(\frac{x-\lambda}{a}\right)^2 + b\left(\frac{x-\lambda}{a}\right)^4\right)^m}\exp\nu\left(-\tan^{-1}\left(\frac{x-\lambda}{a}\right)\right), \tag{19}$$

where λ is the location parameter, $a > 0$ is the scale parameter, $m > \frac{1}{2}$ and $b \geq 0$ control the kurtosis, ν is the asymmetry parameter, and C is the normalization constant. As pointed by Stavroyiannis (2014), the above distribution with the pdf (19) includes an extra fourth order term in the denominator to account for fat and thick-tails for the case of $b > 0$. The distribution becomes double peaked for the case of a negative b coefficient, while for $b = 0$ the Pearson-IV distribution is regained. For details on these, the interested readers are referred to Stavroyiannis (2014).

7. A New Class of Generalized Pearson Distribution arising from Michaelis-Menten Function

Recently, Shakil and Singh (2015) have developed a new class of generalized Pearson distribution arising from Michaelis-Menten Function, which is described below. For details see Shakil and Singh (2015).

For a positive continuous random variable X, we define a new class of generalized Pearson distributions based on the following differential equation

$$\frac{1}{f(x)}\frac{df\;(x)}{dx}=-\frac{a_1+b_1x}{a_2+b_2x}, \qquad (20)$$

which is a special case of the GPE (2) when $m=1,\, n=1$, $a_2=a_3=\cdots=a_m=0$, $b_2=b_3=\cdots=b_n=0$, and where the right hand expression, $\frac{a_1+b_1x}{a_2+b_2x}$, i.e., the ratio of two polynomials of first degree in x, with $a_1,b_1\ge 0;\, a_2,b_2>0;\, a_1b_2\ne a_2b_1$ (except when $a_1,b_1=0$), is known as Michaelis-Menten function. The solution to the differential equation (20) is easily obtained as follows

$$f_X(x)=C\,(1+\alpha\,x)^{\mu}\,(\gamma+\delta\,x)^{-\theta}\,e^{-\lambda x},\ \alpha,\gamma,\delta>0;\,\mu,\theta,\lambda\ge 0;\,x>0, \qquad (21)$$

where $\mu=\frac{a_2\,b_1}{b_2{}^2},\ \alpha=\frac{b_2}{a_2},\ \gamma=a_2,\ \delta=b_2,\ \theta=\frac{a_1}{b_2},\ \lambda=\frac{b_1}{b_2},\ (a_2,b_2\ne 0)$, and C denotes the normalizing constant. In order that the right side of the Eq. (21) represents a probability density function (pdf), we must have

$$\int_0^{\infty} f_X(x)\,dx=\int_0^{\infty} C\,(1+\alpha\,x)^{\mu}\,(\gamma+\delta\,x)^{-\theta}\,e^{-\lambda x}\,dx=1. \qquad (22)$$

(i) In Eq. (22), using twice the binomial series representation $(1+w)^{-s}=\sum_{k=0}^{\infty}\frac{(s)_k\,(-w)^k}{k!}$, for any real value of s, and Eq. 3.351.3/ P. 310 of Gradshteyn and Ryzhik (1980), where $(s)_k=\frac{\Gamma(s+k)}{\Gamma(s)}=s\,(s+1)\ldots(s+k-1),\ (s\ne 0)$, and $(s)_0=1$, denote the Pochhammer symbol, the normalizing constant C is easily given by

$$C=\left[\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^k\,(\theta)_k\,(\mu)_j\,\gamma^{-\theta}\left(\frac{\delta}{\gamma}\right)^k\alpha^j\,\lambda^{-k-j-1}\,(k+j)!}{k!\,j!}\right]^{-1}. \qquad (23)$$

(ii) Again, in Eq. (22), using the binomial series representation

$(1+w)^{-s} = \sum_{k=0}^{\infty} \frac{(s)_k (-w)^k}{k!}$, for any real value of s, and Equation 2.3.6.9

of Prudnikov et al., Vol. 1 (1986), the expression for the normalizing constant C is easily obtained, after simplification, as follows

$$C = \left[\sum_{j=0}^{\infty} \delta^{-\theta} \alpha^{j} (\mu)_j \left(\frac{\gamma}{\delta} \right)^{j+1-\theta} \Psi\left(j+1,\, j+2-\theta;\, \frac{\lambda \gamma}{\delta} \right) \right]^{-1}, \quad (24)$$

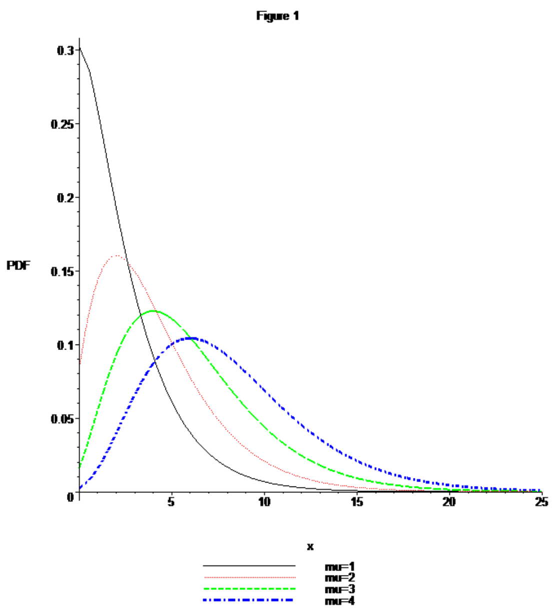
where $\Psi(p, q; z) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-zt} t^{p-1} (1+p)^{q-p-1} \, dt$ is known as

Kummer's (or degenerate hypergeometric) function of the second kind, see, for example, Abramowitz and Stegun (1970), Gradshteyn and Ryzhik (1980), and Oldham et al. (2009), among others.

Using twice the binomial series representation $(1+w)^{-s} = \sum_{k=0}^{\infty} \frac{(s)_k (-w)^k}{k!}$, for any real value of s, and Eq. 3.381.1/P. 317 of Gradshteyn and Ryzhik (1980), the cumulative distribution function (cdf) of our new distribution is easily obtained as follows

$$F_X(x) = \int_0^x f_X(x)\, dx = C \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^k (\theta)_k (\mu)_j \gamma^{-\theta} \left(\frac{\delta}{\gamma} \right)^k \alpha^j \lambda^{-k-j-1} \gamma(k+j+1, \lambda x)}{k!\, j!} \quad (25)$$

where $\gamma(s, z) = \int_0^z t^{s-1} e^{-t} dt$ denotes the incomplete gamma function, and C denote the normalizing constant given by the equation (23). The possible shapes of the pdf $f(x)$ in Eq. (21) and the cdf $F(x)$ in Eq. (23) are provided for some selected values of the parameters in the following Figures 1-2. The effects of parameters can be easily seen from these graphs. Also, it is clear from these graphs that our proposed distributions of the random variable X are positively (that is, right) skewed and unimodal.



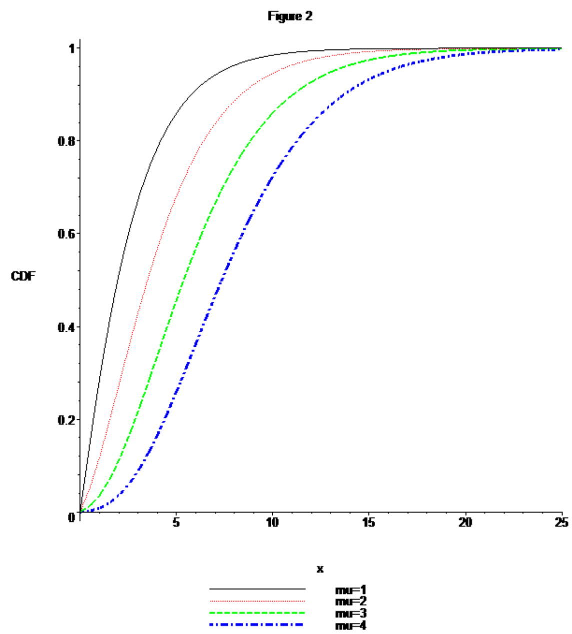


Figure 1: PDF and Figure 2: CDF
for $\mu = 1, 2, 3, 4$ when $\alpha = 1,\ \gamma = 1,\ \delta = 1,\ \theta = 0.5,\ \lambda = 0.5$.

nth **Moment:** It is given by

$$E\left(X^n\right)=\int_0^\infty x^n f_X(x)\,dx = C\int_0^\infty x^n (1+\alpha x)^{\mu}(\gamma+\delta x)^{-\theta} e^{-\lambda x}\,dx. \quad (26)$$

In Eq. (26), using the binomial series representation $(1+w)^{-s}=\sum_{k=0}^{\infty}\frac{(s)_k(-w)^k}{k!}$, for any real value of s, and Eq. 2.3.6.9 of Prudnikov et al., Vol. 1 (1986), the following expression for the nth moment is easily obtained:

$$E\left(X^n\right)=C\sum_{j=0}^{\infty}\frac{\delta^{-\theta}\alpha^j(\mu)_j\left(\frac{\delta}{\gamma}\right)^{n+j+1-\theta}\Gamma(n+j+1)\Psi\left(n+j+1, n+j+2-\theta; \frac{\lambda\delta}{\gamma}\right)}{j!}, \quad (27)$$

where C denotes the normalizing constant given by (24), $\Psi(p, q; z)=\frac{1}{\Gamma(p)}\int_0^\infty e^{-zt}\,t^{p-1}(1+p)^{q-p-1}\,dt$ is known as Kummer's (or degenerate hypergeometric) function of the second kind, and $\Gamma(.)$ denotes the gamma function defined by $\Gamma(s)=\int_0^\infty t^{s-1}e^{-t}dt$, see, for example, Abramowitz and Stegun (1970), Gradshteyn and Ryzhik (1980), and Oldham et al. (2009), among others. Taking $n=1, 2, 3, \ldots,$ in Eq. (27), we can easily obtain the moments of different orders, including the variance, σ^2, of our proposed distribution which can be obtained by using the formula: $\sigma^2=E\left(X^2\right)-\left(E(X)\right)^2$.

Distributional Relationships: It is easy to see that, by a simple transformation of the variable x or by taking special values of the parameters $\{\alpha, \gamma, \delta>0; \mu, \theta, \lambda\geq 0\}$, a number distributions are special cases of Shakil and Singh (2015) distribution as stated below.

(i) Pearson III Distribution (when $\theta=0$).

(ii) Pearson VIII Distribution (when $\mu=0$, $\lambda=0$).

(iii) Pearson IX Distribution (when $\lambda=0$, $\theta=0$).

(iv) Pearson X Distribution (when $\mu = 0$, $\theta = 0$).

(v) A Special Case of Our Proposed Distribution (when $\mu = 0$): When $\mu = 0$ in (21), we have

$$f_X(x) = C\,(\gamma + \delta x)^{-\theta} e^{-\lambda x},\ \alpha, \gamma, \delta > 0; \theta, \lambda \geq 0; x > 0,$$

where C denotes the normalizing constant given by

$$C = \left(\frac{\gamma}{\delta}\right)^{\theta} \left[\gamma\, \Psi\left(1, 2-\theta; \frac{\lambda\gamma}{\delta}\right)\right]^{-1},$$

which is easily obtained by using Equation 2.3.6.9 of Prudnikov et al., Vol. 1 [19], where $\Psi(p, q; z) = \frac{1}{\Gamma(p)} \int_0^{\infty} e^{-zt} t^{p-1} (1+p)^{q-p-1} dt$ is known as Kummer's (or degenerate hypergeometric) function of the second kind, see, for example, Abramowitz and Stegun (1970), Gradshteyn and Ryzhik (1980), and Oldham et al. (2009), among others.

(vi) Distribution of the Product of the PDF's of the Exponential and Some Members of the Family of Burr Distributions (Lomax, or Pareto Type I, or Pareto Type II): It is easy to see that, by a simple transformation of the variable x or by taking special values of the parameters $\{\alpha, \gamma, \delta > 0; \theta, \lambda \geq 0\}$, the pdf of the above special case (v) can be expressed as the pdf of the product of the pdf's of the exponential and some members of the family of Burr distributions (such as Lomax, or Pareto Type I, or Pareto Type II distributions).

8. Concluding Remarks and Directions for Future Research

In this paper, we have reviewed some new classes of continuous probability distribution which can be generated from the generalized Pearson system of differential equation. We have identified as many as fifteen such distributions. It is hoped that the proposed attempt will be helpful in designing a new approach of unifying different families of distributions based on the generalized Pearson differential equation. The other open problems for future research are following:

- **i)** The estimation of the parameters is very important and necessary for applications of these distributions.
- **ii)** Inferences about the parameters may also be of interest to the researchers.

iii) Characterizations of these distributions.
iv) Can we unify all distributions (known & unknown) through GPD?
v) Can we prove Existence & Uniqueness Theorem of Solutions for GPDE?
vi) Can we establish a Fixed Point Theorem for GPDE?

† *M. Shakil, Ph.D,. Miami Dade College, Hialeah, FL, USA*
‡ *B.M. Golam Kibria, Ph.D,. Florida International University, Miami, FL, USA*
§ *J. N. Singh, Ph.D., Barry University, Miami Shores, FL, USA,*

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