Gradient of Scalar Point Function Leading to Expression for Angle in Cartesian Coordinate Geometry

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Abstract: This paper discloses the vital role played by the gradient of a scalar point function in the derivation of the expression for angle in Cartesian coordinate geometry. Treatments are offered in respect of finding the angle between (i) two given straight lines (planes), and (ii) two given planes (tangent planes).

Introduction

One of the important issues in the study of the traditional two dimensional Cartesian coordinate geometry [1-4] is the problem of finding the expression for the angle between two given straight lines. In three dimensional case, i.e. in three dimensional Cartesian coordinate geometry, one is concerned in finding the expression for the angle between two given planes. Analytical methods are employed for the derivation of the expression for the angle in each of the above cases. The expression of the angle so obtained is then employed to find the condition of parallelism and perpendicularity of the given straight lines (planes).
That the gradient of a scalar point function play a vital role in the procedure of derivation of the expression for the angle in different cases in Cartesian coordinate geometry is of prime concern in this paper. Based on the fundamental concept of gradient of scalar point function, derivations are offered in this paper for finding: (i) the expression for the angle between two given straight lines, (ii) the expression for the angle between two given planes, (iii) the expression for the angle between the tangents drawn at two given points lying on a given curve, (iv) the expression for the angle between the tangent planes drawn through two given points lying on a given surface.
The derivations offered with the application of the fundamental concept of gradient in the aforesaid context are novel, general, simple and straight forward. As a result, they will enrich the long-running literature there by increasing the range of applicability of vector algebra and vector calculus as well.

Preliminaries

Dot product: If A and B are two vectors, their dot product denoted by **A**. **B**, is defined as **A**. **B** = |A| |B| $\cos\theta$, where θ is the angle between **A** and **B** such that $0^0 \leq \theta \leq 180^0$.

Scalar point function: If corresponding to each and every point in a certain region of space there exists a scalar φ, then φ is said to be a scalar point function and we write it as $\varphi(x, y, z)$. For example, the temperature at different points in the atmosphere has got well defined values. Thus in this case, the temperature (T) may be considered as a scalar point function. i.e. T = T(x, y, z).

Gradient of a scalar point function: If $\varphi(x, y, z)$ be defined and differentiable at each point in a certain region of space, then the gradient of φ (i.e. grad φ) is defined as, grad φ = $\mathbf{i}\,\frac{\partial}{\partial x}+\mathbf{j}\,\frac{\partial}{\partial y}+\mathbf{k}\,\frac{\partial}{\partial z}$, where **i**, **j**, and **k** are rectangular unit vectors. Now, let $\varphi(x, y, z) = c$, where c is a constant, be a level surface, i.e. a surface corresponding to each point of which the scalar point function φ has got the same constant value c. Let us now consider any point P on this level surface defined by the position vector **r**.

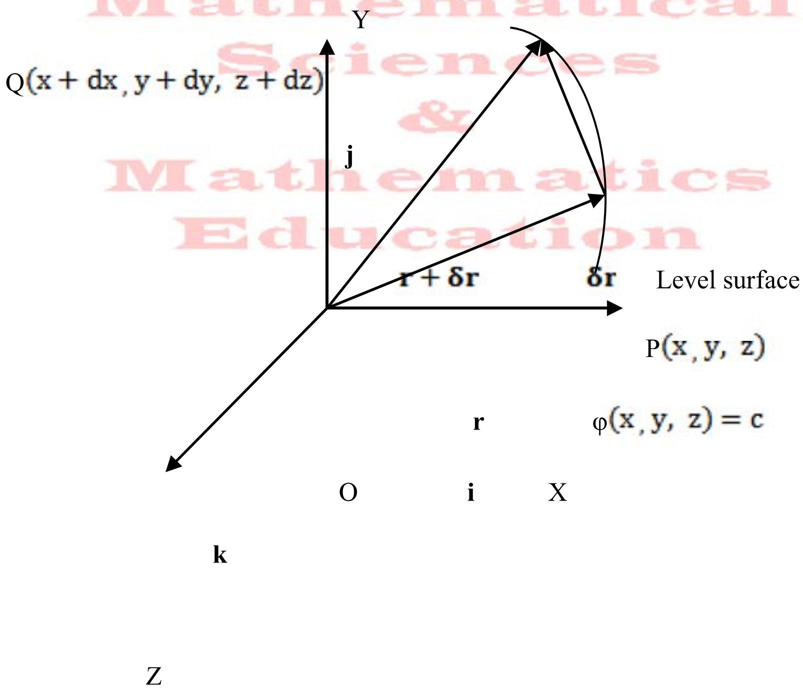

Figure 1. Diagram for finding the direction of gradient of a scalar point function

The position vector of a point Q very close to the point P and lying on the same level surface may then be taken as, $\mathbf{r} + \delta\mathbf{r}$. If the Cartesian coordinates of the point P are (x, y, z), then the Cartesian coordinates of the point Q may be considered as $(x + dx, y + dy, z + dz)$. From Figure 1, we have,

$\delta\mathbf{r} = \mathbf{OQ} - \mathbf{OP}$. Now, we have, (grad φ) . **δr** =

$$\left(\mathbf{i}\,\frac{\partial\varphi}{\partial x}+\mathbf{j}\,\frac{\partial\varphi}{\partial y}+\mathbf{k}\,\frac{\partial\varphi}{\partial z}\right)\,.\,(\mathbf{i}\,dx+\mathbf{j}\,dy+\mathbf{k}\,dz) = \frac{\partial\varphi}{\partial x}\,dx+\frac{\partial\varphi}{\partial y}\,dy+\frac{\partial\varphi}{\partial z}\,dz = d\varphi.$$

Now, since both the points P and Q lie on the same level surface defined by $\varphi(x, y, z) = c$, the value of φ at each of the points P and Q must be each equal to c. Thus we must have, $d\varphi = 0$ in this case. It then follows from the aforesaid relation that, (grad φ). $\boldsymbol{\delta r} = 0$.

Thus grad φ is a vector perpendicular to $\boldsymbol{\delta r}$, now since the points P and Q are very close to each other, $\boldsymbol{\delta r}$ may be assumed to lie on the level surface under consideration. Thus grad φ is a vector which is normal to the level surface $\varphi(x, y, z) = c$ at the point (x, y, z).

With a view to enhancing the range of applicability of the aforesaid concept of gradient of scalar point function, efforts have been made in the subsequent sections for the generation of the expressions for the angle between two given straight lines as well as that between two given planes.

Derivation of the Expression for Angle in Cartesian Coordinate Geometry on the Basis of Gradient of Scalar Point Function

●To find the angle between two given straight lines

As shown in Figure1, let the equations of the two given straight lines PQ and RS be $y = m_1 x + c_1$ and $y = m_2 x + c_2$ respectively. It is required to find the angle θ (as shown in Figure 2) between those two straight lines.

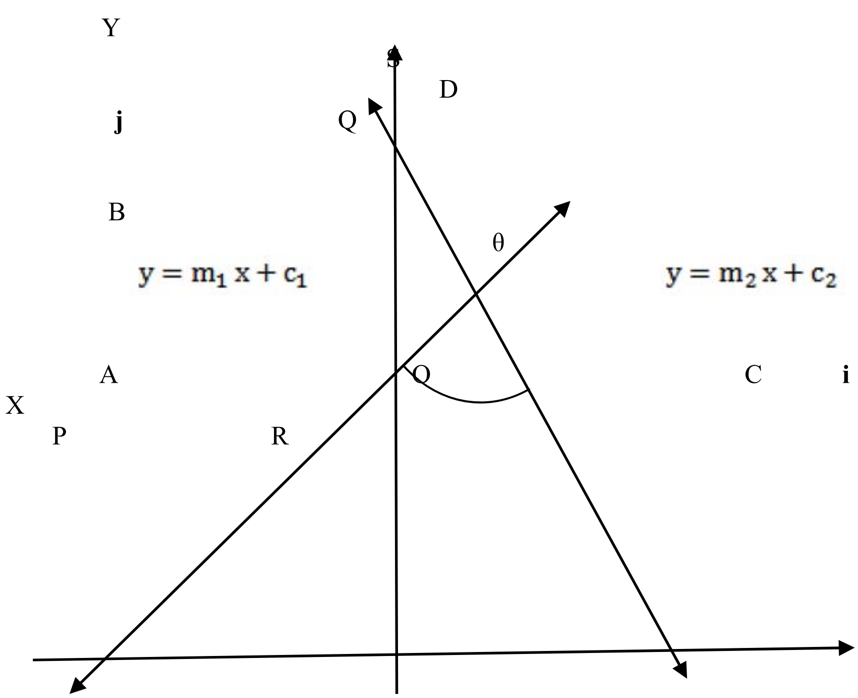

Figure 2. Diagram for finding the angle between two given straight lines

From simple geometrical consideration, it can be readily seen that the angle between the two normal vectors to two straight lines is equal to one of the two angles between the straight lines. This fundamental assertion can be employed to find the angle between the two given straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$.

The equations of the two given straight lines may be written as $\varphi_1(x, y) = m_1 x - y + c_1 = 0$, and $\varphi_2(x, y) = m_2 x - y + c_2 = 0$, where $\varphi_1(x, y)$ and $\varphi_2(x, y)$ are scalar point functions. Let $\mathbf{n_1}$ and $\mathbf{n_2}$ be respectively the unit normal vectors at any point (x, y) of the two given straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$. then we have,

$\mathbf{n_1} = \frac{\{\text{grad } \varphi_1 (x,y)\}}{|\{\text{grad } \varphi_1 (x,y)\}|} = (m_1 \mathbf{i} - \mathbf{j})/\sqrt{(1 + m_1{}^2)}$,

and $\mathbf{n_2} = \frac{\{\text{grad } \varphi_2 (x,y)\}}{|\{\text{grad } \varphi_2 (x,y)\}|} = (m_2 \mathbf{i} - \mathbf{j})/\sqrt{(1 + m_2{}^2)}$.

Thus we have, $\mathbf{n_1} . \mathbf{n_2} = (1 + m_1 m_2)/\sqrt{\{(1 + m_1{}^2)(1 + m_2{}^2)\}}$

or, $\cos\theta = \frac{1 + m_1 m_2}{\sqrt{(1+m_1{}^2)(1+m_2{}^2)}}$ … (1), where θ being the angle between the two normal vectors $\mathbf{n_1}$ and $\mathbf{n_2}$ is also equal to one of the two angles between the two given straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$.

Condition of parallelism of two given straight lines

When the straight lines PQ and RS represented respectively by the equations $y = m_1 x + c_1$ and $y = m_2 x + c_2$ are parallel to each other, then the angle θ between the vectors, **AB** and **CD** will be either 0^0 or 180^0 depending on whether the vectors, **AB** and **CD** are like parallel or unlike parallel.

Now, when $\theta = 0^0$, we have from equation (1),

$1 = \frac{1 + m_1 m_2}{\sqrt{(1+m_1{}^2)(1+m_2{}^2)}}$,

which on simplification gives, $m_1 = m_2$.

Again when $\theta = 180^0$, equation (1) gives

$-1 = \frac{1 + m_1 m_2}{\sqrt{(1+m_1{}^2)(1+m_2{}^2)}}$,

which on simplification yields the same relation, $m_1 = m_2$.

Thus the condition of parallelism of the two given straight lines $y = m_1 x + c_1$ and $y = m_2 x + c_2$ is given by $m_1 = m_2$.

Condition of perpendicularity of two given straight lines

When the straight lines PQ and RS represented respectively by the equations, $y = m_1 x + c_1$ and $y = m_2 x + c_2$ are perpendicular to each other, then the angle θ between the vectors, **AB** and **CD** will be 90^0. Under this condition, the equation (1) gives,

$0 = \frac{1+m_1 m_2}{\sqrt{(1+m_1^2)(1+m_2^2)}}$,

from which it follows that $m_1 \; m_2 = -1$.

Thus the condition that the straight line $y = m_1 x + c_1$ is perpendicular to the straight line $y = m_2 x + c_2$ is given by $m_1 \; m_2 = -1$.

●To find the angle between two given planes

Let the equations of the two given planes be $a_1 x + b_1 y + c_1 z + d_1 = 0$, and $a_2 x + b_2 y + c_2 z + d_2 = 0$. It is required to find the angle between these two given planes. Now, the equations of the two given planes may be written as $\varphi_1(x, y, z) = 0$, and $\varphi_2 (x, y, z) = 0$ respectively where $\varphi_1(x, y, z) = a_1 x + b_1 y + c_1 z + d_1$, and $\varphi_2 (x, y, z) = a_2 x + b_2 y + c_2 z + d_2$ are scalar point functions.

If $\mathbf{n_1}$ and $\mathbf{n_2}$ are respectively the unit normal vectors to the aforesaid given planes $\varphi_1(x, y, z) = 0$, and $\varphi_2 (x, y, z) = 0$, then from the physical meaning of the gradient of a scalar point function, we must have,

$\mathbf{n_1} = \frac{\{\text{grad}\, \varphi_1 (x,y,z)\}}{|\{\text{grad}\, \varphi_1 (x,y,z)\}|} = \frac{\mathbf{i}\, a_1 + \mathbf{j}\, b_1 + \mathbf{k}\, c_1}{\sqrt{(a_1^{\,2} + b_1^{\,2} + c_1^{\,2})}}$

and $\mathbf{n_2} = \frac{\{\text{grad}\, \varphi_2 (x,y,z)\}}{|\{\text{grad}\, \varphi_2 (x,y,z)\}|} = \frac{\mathbf{i}\, a_2 + \mathbf{j}\, b_2 + \mathbf{k}\, c_2}{\sqrt{(a_2^{\,2} + b_2^{\,2} + c_2^{\,2})}}$

Then, $\mathbf{n_1} . \mathbf{n_2} = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left\{\sqrt{(a_1^{\,2} + b_1^{\,2} + c_1^{\,2})(a_2^{\,2} + b_2^{\,2} + c_2^{\,2})}\right\}}$

or, $\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left\{\sqrt{(a_1^{\,2} + b_1^{\,2} + c_1^{\,2})(a_2^{\,2} + b_2^{\,2} + c_2^{\,2})}\right\}}$, where θ is the angle between the two unit normal vectors to the two given planes respectively.

Now, making use of the theoretical assertion that the angle between the normal vectors to two different planes is equal to one of the two angles between the planes, it may be said that here θ is also one of the angles between the two given planes $a_1 x + b_1 y + c_1 z + d_1 = 0$, and $a_2 x + b_2 y + c_2 z + d_2 = 0$.

Hence the angle between the two given planes $a_1 x + b_1 y + c_1 z + d_1 = 0$, and $a_2 x b_2 y + c_2 z + d_2 = 0$, must be given by the relation

$$\cos\theta = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left\{\sqrt{(a_1^{\,2} + b_1^{\,2} + c_1^{\,2})(a_2^{\,2} + b_2^{\,2} + c_2^{\,2})}\right\}} \qquad (2)$$

Condition of parallelism of two given planes

If the two given planes $a_1 x + b_1 y + c_1 z + d_1 = 0$, and $a_2 x + b_2 y + c_2 z + d_2 = 0$ are parallel, then the angle θ between the unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ to those given planes must be either 0^0 or 180^0. Thus under this condition, we have from the relation (2),

$$\pm 1 = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left\{\sqrt{(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2)}\right\}} \qquad \ldots \qquad (3)$$

Squaring both sides of the relation (3) and then rearranging we then obtain, $(a_1 b_2 - a_2 b_1)^2 + (a_1 c_2 - a_2 c_1)^2 + (b_1 c_2 - b_2 c_1)^2 = 0$, from which it then readily follows that, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Thus the condition that the plane $a_1 x + b_1 y + c_1 z + d_1 = 0$ is parallel to the plane $a_2 x + b_2 y + c_2 z + d_2 = 0$ is given by, $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$.

Condition of perpendicularity of two given planes

If the plane $a_1 x + b_1 y + c_1 z + d_1 = 0$ is perpendicular to the plane $a_2 x + b_2 y + c_2 z + d_2 = 0$, then the angle θ between the two unit normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ to those given planes must be 90^0. Under this condition, we have from the relation (2),

$$0 = \frac{a_1 a_2 + b_1 b_2 + c_1 c_2}{\left\{\sqrt{(a_1{}^2 + b_1{}^2 + c_1{}^2)(a_2{}^2 + b_2{}^2 + c_2{}^2)}\right\}},$$

from which it readily follows that, $a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$

Thus the condition that the plane $a_1 x + b_1 y + c_1 z + d_1 = 0$ is perpendicular to the plane $a_2 x + b_2 y + c_2 z + d_2 = 0$ is given by,

$a_1 a_2 + b_1 b_2 + c_1 c_2 = 0.$

● To find the angle between the tangents drawn at two given points lying on a given curve

Let the given curve be a circle whose equation is $x^2 + y^2 = a^2$ and also let the two given points lying on this circle be (x_1, y_1) and (x_2, y_2). It is required to find the angle (θ) between the tangents drawn to the circle $x^2 + y^2 = a^2$ at the points (x_1, y_1) and (x_2, y_2).

Here, the equation of the circle may be written as $\varphi(x, y) = x^2 + y^2 - a^2 = 0$, where $\varphi(x, y)$ is a scalar point function. Then we have, grad $\varphi = 2x\,\mathbf{i} + 2y\,\mathbf{j}$. Now, if $\mathbf{n}_1$ and $\mathbf{n}_2$ are respectively the unit normal vectors to the circle $x^2 + y^2 = a^2$ at the points (x_1, y_1) and (x_2, y_2), we have,

$$\mathbf{n}_1 = \frac{\{(\text{grad}\,\varphi)_{(x_1, y_1)}\}}{\{|(\text{grad}\,\varphi)_{(x_1, y_1)}|\}} = \frac{x_1}{a}\,\mathbf{i} + \frac{y_1}{a}\,\mathbf{j},$$

and $\mathbf{n_2} = \frac{\{(\text{grad}\,\varphi)_{(x_2,\,y_2)}\}}{\{|(\text{grad}\,\varphi)_{(x_2,\,y_2)}|\}} = \frac{x_2}{a}\,\mathbf{i} + \frac{y_2}{a}\,\mathbf{j}$.

Then, $\cos\theta = \mathbf{n_1}\,.\,\mathbf{n_2}$

or, $\cos\theta = (\frac{x_1}{a}\,\mathbf{i} + \frac{y_1}{a}\,\mathbf{j})\,.\,(\frac{x_2}{a}\,\mathbf{i} + \frac{y_2}{a}\,\mathbf{j})$

or, $\cos\theta = \frac{x_1 x_2 + y_1 y_2}{a^2}$ … (4)

The relation (4) gives the expression for the angle (θ) between the tangents drawn at the given points (x_1, y_1) and (x_2, y_2) lying on the given circle $x^2 + y^2 = a^2$.

Let us now consider the given curve as the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Also let the two given points lying on this circle be (x_1, y_1) and (x_2, y_2). It is required to find the angle (θ) between the tangents drawn to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the points (x_1, y_1) and (x_2, y_2).

Here, the equation of the circle may be written as $\varphi(x, y) = x^2 + y^2 + 2gx + 2fy + c = 0$, where $\varphi(x, y)$ is a scalar point function.

Then we have, grad $\varphi = 2(x+g)\,\mathbf{i} + 2(y+f)\,\mathbf{j}$. Now, if $\mathbf{n_1}$ and $\mathbf{n_2}$ are respectively the unit normal vectors to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ at the points (x_1, y_1) and (x_2, y_2), we have,

$\mathbf{n_1} = \frac{\{(\text{grad}\,\varphi)_{(x_1,\,y_1)}\}}{\{|(\text{grad}\,\varphi)_{(x_1,\,y_1)}|\}} = \frac{\{(x_1+g)\,\mathbf{i} + (y_1+f)\,\mathbf{j}\}}{\sqrt{\{(x_1+g)^2 + (y_1+f)^2\}}}$, and $\mathbf{n_2} = \frac{\{(\text{grad}\,\varphi)_{(x_2,\,y_2)}\}}{\{|(\text{grad}\,\varphi)_{(x_2,\,y_2)}|\}} = \frac{\{(x_2+g)\,\mathbf{i} + (y_2+f)\,\mathbf{j}\}}{\sqrt{\{(x_2+g)^2 + (y_2+f)^2\}}}$.

Then we have,

$\cos\theta = \mathbf{n_1}\,.\,\mathbf{n_2}$

or, $\cos\theta = \frac{[\{(x_1+g)\,\mathbf{i} + (y_1+f)\,\mathbf{j}\}.\,\{(x_2+g)\,\mathbf{i} + (y_2+f)\,\mathbf{j}\}]}{[\sqrt{\{(x_1+g)^2 + (y_1+f)^2}\sqrt{\{(x_2+g)^2 + (y_2+f)^2\}}]}$

or, $\cos\theta = \frac{\{(x_1+g)(x_2+g) + (y_1+f)(y_2+f)\}}{[\sqrt{\{(x_1+g)^2 + (y_1+f)^2}\sqrt{\{(x_2+g)^2 + (y_2+f)^2\}}]}$ (5)

The relation (5) is the expression for the angle (θ) between the tangents drawn at the given points (x_1, y_1) and (x_2, y_2) lying of the given circle $x^2 + y^2 + 2gx + 2fy + c = 0$.

Proceeding similarly, it is possible to find the expression for the angle between the tangents drawn at two given points lying on a given curve irrespective of its nature.

● To find the angle between the tangent planes drawn through two given points lying on a given surface

Let the given surface be a sphere whose equation is $x^2 + y^2 + z^2 = a^2$ and also let the two given points lying on this sphere be (x_1, y_1, z_1) and (x_2, y_2, z_2). It is required to find the angle (θ) between the tangent planes drawn to the sphere $x^2 + y^2 + z^2 = a^2$ at the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

Here, the equation of the sphere may be written as $\varphi(x, y) = x^2 + y^2 + z^2 - a^2 = 0$, where $\varphi(x, y)$ is a scalar point function. Then we have, grad $\varphi = 2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}$. Now, if $\mathbf{n_1}$ and $\mathbf{n_2}$ are respectively the unit normal vectors to the sphere $x^2 + y^2 + z^2 = a^2$ at the points (x_1, y_1, z_1) and (x_2, y_2, z_2), we have,

$$\mathbf{n_1} = \frac{\{(\text{grad}\,\varphi)_{(x_1, y_1, z_1)}\}}{\{\left|(\text{grad}\,\varphi)_{(x_1, y_1, z_1)}\right|\}} = \frac{x_1}{a}\mathbf{i} + \frac{y_1}{a}\mathbf{j} + \frac{z_1}{a}\mathbf{k},\ \mathbf{n_2} = \frac{\{(\text{grad}\,\varphi)_{(x_2, y_2, z_2)}\}}{\{\left|(\text{grad}\,\varphi)_{(x_2, y_2, z_2)}\right|\}} = \frac{x_2}{a}\mathbf{i} + \frac{y_2}{a}\mathbf{j} + \frac{z_2}{a}\mathbf{k}.$$

Then, $\cos\theta = \mathbf{n_1} \cdot \mathbf{n_2}$

or, $\cos\theta = \left(\frac{x_1}{a}\mathbf{i} + \frac{y_1}{a}\mathbf{j} + \frac{z_1}{a}\mathbf{k}\right) \cdot \left(\frac{x_2}{a}\mathbf{i} + \frac{y_2}{a}\mathbf{j} + \frac{z_2}{a}\mathbf{k}\right)$

or, $\cos\theta = \frac{x_1 x_2 + y_1 y_2 + z_1 z_2}{a^2}$ … (6)

This relation (6) gives the expression for the angle between the two tangent planes passing through the given points (x_1, y_1, z_1) and (x_2, y_2, z_2) lying on the surface of the given sphere $x^2 + y^2 + z^2 = a^2$.

Proceeding similarly, it is possible to derive the expression for the angle between the tangent planes passing through two given points lying on a given surface irrespective of its nature.

Conclusion

With a view to increasing the range of applicability of vector algebra as well as vector calculus, an attempt has been made in this paper to derive the expressions for the angle between two given straight lines(planes), that between the tangents (tangent planes) drawn at two given points lying on a given curve (surface) by making use of the fundamental concept of gradient of a scalar point function. It has been observed that the gradient of a scalar point function [5] plays a vital role in the treatments of derivation offered. The techniques offered are novel, simple, straight forward and generalized ones on account of the fact that they are equally applicable for any given curve, irrespective of its nature. As a result, they will enrich the traditional literature there by enhancing the same as well.

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