A Brief History of the Method of Exhaustion with an Illustration

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Abstract

In this paper, we discuss the contributions of Antiphon of Athens, Bryson of Heraclea, and Eudoxus to the method of exhaustion, a forerunner of the integral calculus. We illustrate how the method was used by Eudoxus to prove that the areas of two circles are in the same ratio as the squares of their diameters, as presented in Euclid's proposition XII.2.

Introduction

As one examines Euclid's Elements Book XII, one notices that the method used to establish the veracity of some of the propositions contained in such a book is quite different from the techniques used in previous books. The main feature of the new method involves a limiting-like process, known today as the method of exhaustion. This process is considered as a precursor of integral calculus.

In this paper, we provide a brief history of the method of exhaustion and we illustrated it with its first use, the proof of Euclid's proposition XII.2: Circles are to one another as the squares on their diameters. Archimedes attributed this proof to Eudoxus (Heath, 1956).

Description of the Method of Exhaustion

The method of exhaustion is a logical technique developed by the classical Greeks to prove theorems involving areas curvilinear plane figures as well as volumes of solids. The method consisted of inscribing, and sometimes circumscribing, in a shape a sequence of geometric figures whose property in question converges to the property of the shape to be investigated. In particular, to prove a proposition about the area of a shape, was inscribe (and sometimes circumscribe) in it (around it) a sequence of polygons such that the differences in area between the nth polygon and the geometric figure can be made as small as possible by choosing n sufficiently large (Figure 1).

Often, the method of exhaustion involves using a "*double reductio ad absurdum*" argument. This indirect approach uses the trichotomy law: Given two arbitrary quantities a and b, one and only one of the following relations holds: $a < b$, $a = b$, or $a > b$). To prove that $a = b$ using *double reductio ad absurdum*, we first suppose that $a < b$ and derive a logical contradiction, thereby eliminating the possibility that $a < b$. We then assume that $a > b$ and, once again, arrive at a logical contradiction, excluding the option that $a > b$. Once we eliminate these two cases, we must irrefutably conclude that $a = b$.

The term "method of exhaustion" was first used in 1647 by the Jesuit mathematician Gregory of Saint Vincent in his *Opus Geometricum Quadraturae Circuli Sectionum Coni* (Geometric work on the quadrature of the circle of conic sections)

As with most mathematical ideas, the method of exhaustion is the product and refinement of several mathematicians, notably Antiphon of Athens, Bryson of Heraclea, and Eudoxus of Cnidus. However, no one else use the method of exhaustion more deftly than Archimedes. The central idea of the method of exhaustion seems to have originated with Antiphon of Athens, also known as Antiphon the sophist (O'Connor & Roberston, 1999).

Antiphon of Athens

Antiphon of Athens (480 BC – 411 BC), a contemporary of Socrates, was a Greek orator with some interests in mathematics. He is credited as being the first person who proposed that the area of a circle be calculated in terms of the area of inscribed regular polygons (Anglin, 1994). Figure 1 shows several of the constructible regular polygons inscribed in a circle. As the number of sides increases, we notice that the area of the inscribed polygons approaches the area of the circle. Also, Antiphon observed that an inscribed square covers more than 1/2 the area of the circle while an octagon covers more than 3/4 the area of a circle. In general, Antiphon realized that an inscribed regular 2^n polygon covers more than $(1 - \frac{1}{2^{n-1}})$ of the area of the circle (Anglin, 1994).

Antiphon is the first known person to have used the method of exhaustion. According to Heath,

> Antiphon therefore deserves an honourable place in the history of geometry as having originated the idea of exhausting an area by means of inscribed regular polygons with an ever increasing number of sides, an idea upon which ... Eudoxus founded his epoch-making method of exhaustion (p. 222).

The next contribution to the development of the method of exhaustion was made by Bryson of Heraclea.

Bryson of Heraclea

Bryson of Heraclea was a Greek sophist mathematician who seems to have been a pupil of Socrates. His methods to square the circle resulted in a further development of mathematics. Specifically, he took a step further than Antiphon by inscribing and circumscribing polygons in and around a circle.

Bryson claimed that the area of a circle was greater than the area of any of the inscribed regular polygons, but less than the area of any of the circumscribed regular polygons (Heath, 1921). It seems likely that Bryson was claiming that taking inscribed and circumscribed polygons with a greater and greater number of sides, the difference between the area of the inscribed and the associated circumscribed polygon could be made as small as we choose. He then seems to claim that this process generates an intermediate polygon whose area differs from that of the circle as little as we want (O'Connor and Robertson, 1999). As noted by Heath (1921), this process is characteristic of the method of exhaustion as masterfully used by Archimedes. The method of exhaustion, however, was formalized by Eudoxus of Cnidus.

Eudoxus of Cnidus

Eudoxus of Cnidus (408 BC – 355 BC) was a Greek scholar who made contributions to astronomy, medicine, geography, philosophy, and mathematics (Anglin, 1994). He made two significant contributions to mathematics. He developed the theory of proportion that possibly allowed comparing irrational lengths, resolving the crisis produced by the discovery of incommensurable quantities. This theory is included in Euclid's Elements Book V. Eudoxus also formalized some of the ides of Antiphon and Bryson by transforming them into a rigorous method known today as the method of exhaustion. Much of Eudoxus's work involving the method of exhaustion is set out in Euclid's Elements book XII. The first proposition in the Elements where the method of exhaustion is used is in proposition XII.2.

Eudoxus's Proof of Euclid's Proposition XII.2: Preliminaries

Euclid's Proposition XII.2 in his own language is as follows: circles are to one another as the squares of their diameters. In current terms, this proposition can be reformulated as follows: the areas of two circles are in the same ratio as the squares of their diameters. To prove this proposition, we need the following four principles/theorems.

1. Axiom of comparability (a.k.a Eudoxus' Axiom or lemma of Archimedes);
2. Principle of exhaustion;
3. Euclid's proposition XII.1: The ratios of the areas of two similar polygons inscribed in circles is equal to the ratio of the squares of their diameters;
4. For any given area, we can inscribe a regular polygon in a given circle such that the difference between the area of the circle and the area of the polygon is less than the given area.

Eudoxus used the axiom of comparability commonly known as "the lemma of Archimedes." Such a lemma is definition 4 in Euclid's Elements Book V:

Magnitudes are said to have a ratio to one another which are capable, when multiplied, of exceeding one another (Boyer, 1970, p. 63).

This definition is really an axiom whose interpretation in modern terms says that the ratio $a : b$ exists if there are numbers n and m such that $na > b$ and $mb > a$ (Artmann, 1999). Notice that this axiom restricts the existence of ratios to magnitudes of the same type. If one magnitude is so small (e.g., infinitesimals) or so large that there is not a multiple of one that exceed the other, then the ratio does not exist.

This axiom of comparability was used by Eudoxus to prove proposition 1 in Euclid's Elements Book X, which is sometimes referred to as the principle of exhaustion:

Two unequal magnitudes are set out, if from the greater there be subtracted a magnitude greater than its half, and from that which is left a magnitude greater than its half, and if this process is repeated continually, there will be left some magnitude which will be less than the lesser magnitude set out (Boyer, 1970, p. 63).

In other words, if we are given two quantities Q and E with Q > E, we start by subtracting Q_1 from Q where $Q_1 \geq Q/2$ leaving a remainder $R_1 = Q - Q_1$. From R_1, we subtract Q_2 where $Q_2 \geq R_1/2$ leaving a remainder $R_2 = R_1 - Q_2$. From R_2, we subtract Q_3 where $Q_3 \geq R_2/2$ leaving a remainder $R_3 = R_2 - Q_3$. Euclid's proposition X.1 guarantees that we can continue this process (a finite number of times) until we have a quantity $R_n < E$. That is,

$R_n = Q - (Q_1 + Q_2 + Q_3 + \ldots + Q_n) < E$ for n sufficiently large and arbitrary E

In our current notation, $Q - (Q_1 + Q_2 + Q_3 + \ldots + Q_n) < E$ for n sufficiently large and arbitrary E means that $Q = \lim_{n \to \infty}(Q_1 + Q_2 + Q_3 + \ldots + Q_n)$. Notice, however, that Eudoxus' process is finite.

As noted by Boyer (1970), the principle of exhaustion can be generalized by "replacing "greater than its half" by "greater or equal to its half" (or its third or any other proper fraction." (p. 63). In particular, this principle guarantees that any quantity Q can be made smaller than a second quantity E by successively halving it a finite number of times.

The proof of Euclid's proposition XII.2 uses proposition XII.1, which he stated as follows: Similar polygons inscribed in circles are to one another as the squares on their diameters.

Finally, we will show that for any given area, we can inscribe a regular polygon in a given circle such that the difference between the area of the circle and the area of the polygon is less than the given area. To this end, let AB be the side of a regular polygon inscribed in a given circle (Figure 2). We construct C, the midpoint of arc ACB. We will show that the sum of the areas of the circular segments AC and CB is less than half the area of the circular segment AB.

Through C, construct a tangent line to the circle. Denote by BD and AE the segments that go through B and D and are perpendicular to the tangent. ∠DCB ≅ ∠CAB because ∠DCB is semi-inscribed in arc CB and ∠CAB is inscribed in arc CB. Now, ∠CAB ≅ ∠CBA because the angles opposite the congruent sides of an isosceles triangle are congruent. Therefore, segment ED || segment AB and hence ABDE is a rectangle. Notice next that the area of triangle ABC is half the area of rectangle ABDE. Observe also that the area of segment AC is less than the area of ΔACE and therefore less than the area of ΔAFC. Similarly, the area of segment CB is less than the area of ΔBFC. Thus, the circular segment AB has been divided into two parts (the triangle ABC and the combined part consisting of circular segments AC and BC), with one less than the other. Therefore, the combined area of circular segments AC and BC (the smaller part) is less than half of the area of the circular segment AB.

Let E represent any area. The area of the circular segment AB is part of the area inside the circle but outside of the inscribed polygon with side AB. It represents the initial quantity Q. From this quantity Q, we subtract the area of ΔABC (the quantity $Q_1 \geq Q/2$) leaving as remainder $R_1 = Q - Q_1$, the combination of the areas of the circular segments AC and BC, which is less than half of the previous quantity. Notice that by replacing side AB by sides AC and BC, we have doubled the number of sides of the regular polygon and reduced by more than half the excluded area (the area inside the circle but outside of the polygon. By continuing this doubling process, the condition stated in the principle of exhaustion is met. Therefore, we can conclude that, by repeatedly doubling the number of sides of a regular polygon, given any area E, we can inscribe a regular polygon in a given circle such that the difference between the area of the circle and the area of the polygon is less than the given area E.

Eudoxus's Proof of Euclid's Proposition XII.2: The Proof

Euclid formulated his proposition XII.2 as follows: Circles are to one another as the squares on their diameters. Using contemporary language, we can reformulate this proposition in the following terms: the ratio of the areas of two circles is equal to the ratio of their squares of their diameters. We paraphrase Euclid's argument in modern notation.

Let C_1 and C_2 be two circles with areas A_1 and A_2 respectively. Denote by $d_1$and d_2 their corresponding diameters. We want to show that $A_1 : A_2 = d_1^2 : d_2^2$. Suppose, to the contrary, that $A_1 : A_2 \neq d_1^2 : d_2^2$. Therefore, there exists a circle C with area S such that $d_1^2 : d_2^2 = A_1 : S$ with $S < A_2$ or $S > A_2$.

Assume that S < A_2. Notice that, by the principle of exhaustion, there exists also a regular polygon P_n with area P inscribed in C_2 such that A_2 – P < A_2 – S or S < P < A_2. There exists a regular polygon Q_n with area Q inscribed in C_1 such that $P_n \sim Q_n$. Invoking the fact that the ratio of the areas of similar polygons inscribed in circles is the ratio of their squares of the diameters, we conclude Q: P= $d_1^2 : d_2^2 = A_1$: S, thereby inferring that Q: P = A_1: S or A_1 : Q = S: P. Because A_1 > Q, we have that S > P, contradicting the fact that S < P. Therefore, S can't be less than A_2.

Next, assume that S > A_2. We can rewrite $d_1^2 : d_2^2 = A_1$: S as $d_2^2 : d_1^2$ = S: A_1. There exists a circle C' with area T such that S: A_1 = A_2: T or S: A_2 = A_1 : T. Because S > A_2, we have A_1 > T or T < A_1. Using exactly the same procedure to show that S < A_2 is impossible, we can prove that T < A_1 is impossible as well. We are forced to infer that S can't be greater than A_2.

Since both S < A_2 and S > A_2 are impossible, we deduct that S = A_2. In other words, A_1: $A_2 = d_1^2 : d_2^2$, as requested.

Notice that rather than using the principle of exhaustion with circumscribing regular polygons instead of inscribed regular polygons, Euclid reduced the second case to the case already proved.

As a corollary to this theorem, we can now show that the area of a circle is proportional to its radius. That is, A = kr^2, where *k* is the area of a circle with radius 1. It was Archimedes who proved that *k* = π.

The Method of Exhaustion and the Limit Process

The example described above (Euclid XII.2) illustrates how the method worked. Notice that the method involves using a "*double reductio ad absurdum*" argument. Second, observe also that the method of exhaustion does not really exhaust the property (area or volume). As the example illustrates, the difference between the area of the circle and the polygon is never zero, and thus the area is never exhausted. Third, notice also that the method of exhaustion is a strictly logical process with no reference to the idea of limit not infinitesimal quantities.

Conclusion

Eudoxus is generally considered the second great mathematicians of antiquity, next to the incomparable Archimedes (Dunham, 1990). He was the first mathematician to use the method of exhaustion as a general technique to rigorously prove theorem whose truths had been informally discovered by other means. For example, proposition XII.2 of Euclid's elements (Circles are to one another as the squares on their diameters) was known by Babylonians and Egyptians more than a millennium before Euclid's time.

We should mention that Euclid uses the method of exhaustion to prove also propositions, XII.5, XI.10, XII.11, XII.12, and XII.18 while Archimedes masterfully used it to prove results that he discovered by other means. However, as the difficulty of the problems increased, the method of exhaustion became cumbersome and the need to create new and more powerful methods led mathematicians to invent the integral calculus.

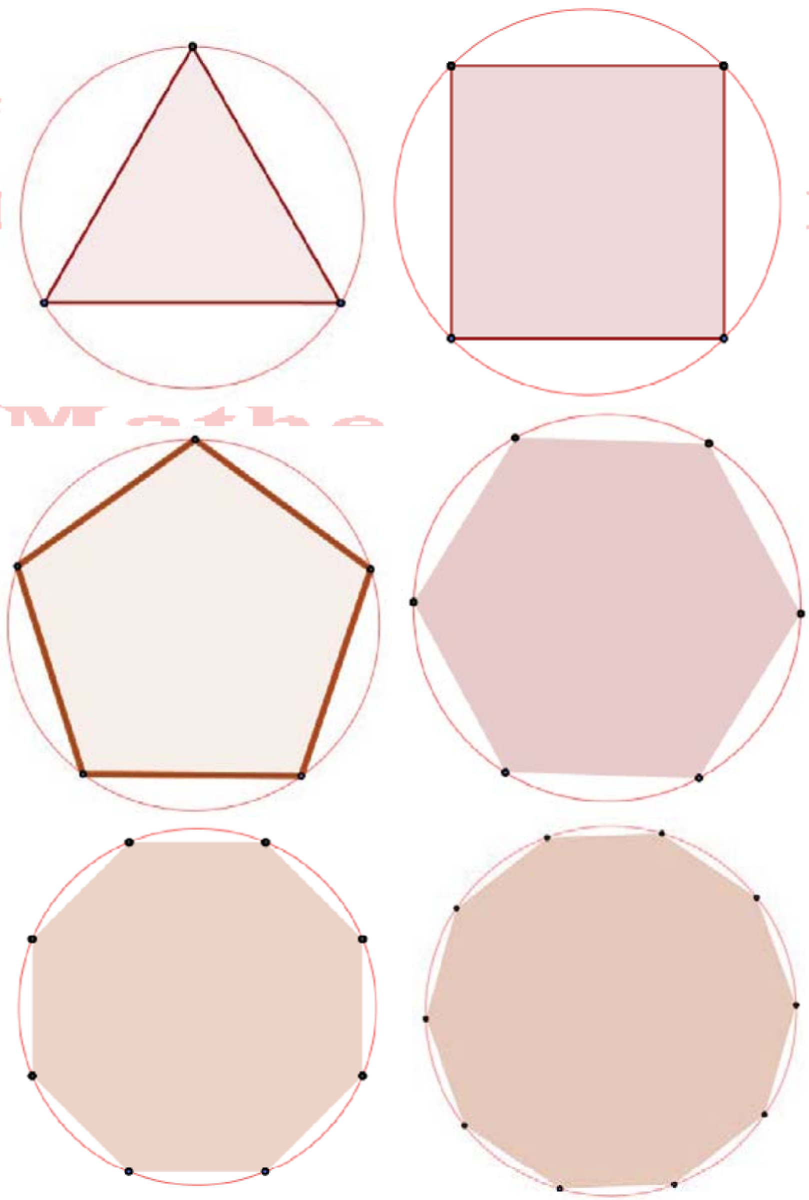

Figure 1: The area of a regular polygon converges to the area of the circle as the number of sides increases.

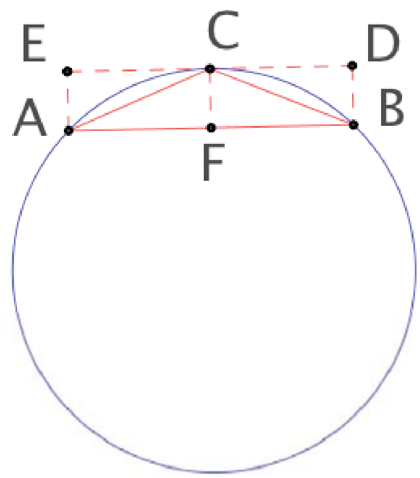

Figure 2: The sum of the areas of circular segments AC and BC is less than half the area of circular segment AB.

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